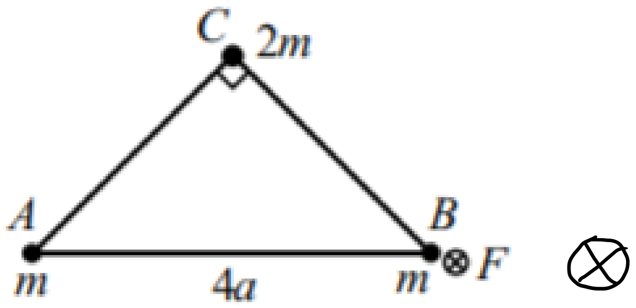
Physics I

Lecture 32

Two classes of problems

1. Strike a rigid object with an impulsive blow and ask what is the motion of the object immediately after the blow.

2. An object rotates about a fixed axis. A given torque is applied to it. What is the frequency of rotation.

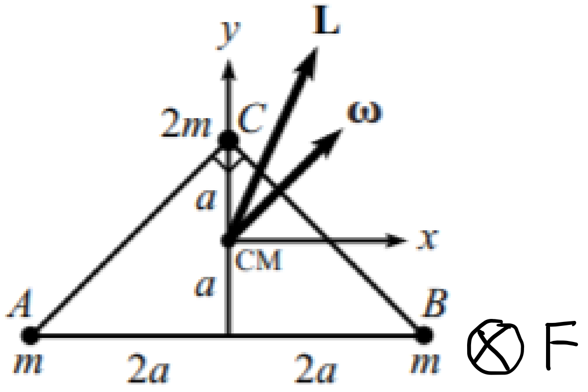


- Three massless rods arranged in an isosceles right angled triangle, with given masses at vertices.
- floating freely in space.
- Impulse mag $\int F\,dt = P$

What are velocities of the 3 masses after the blow.

Strategy

- Find angular momentum relative to CM.
- Identify principal axes and calculate principal moments of inertia
- find $\vec{\omega}$ from $\vec{L} = \{I\}\vec{\omega}$
- find $\vec{v}$ from $\vec{\omega}$.
- Then add on CM motion.



- CM lies at mid pt of altitude.

- CM as origin.

Positions of masses.

$\vec{r}_A = (-2a, -a, 0)$, $\vec{r}_B = (2a, -a, 0)$

$\vec{r}_C = (0, a, 0)$.

x-y plane, z axis out of the plane.

1. Find $\vec{L}$; $\frac{d\vec{L}}{dt} = \vec{\tau}$, $\vec{L} = \int \vec{\tau}\, dt$.

$$\vec{L} = \int \vec{\tau}\, dt = \int \vec{r}_B \times \vec{F}\, dt \quad ; \quad \vec{P} = \int \vec{F}\, dt = (0, 0, -P)$$

$$= \vec{r}_B \times \underbrace{\int \vec{F}\, dt}_{P} \rightarrow$$ [sudden implilse $\vec{r}_B$ stays const]

$$\boxed{\vec{L} = (2a, -a, 0) \times (0, 0 - P) = aP(1, 2, 0)}$$

2. Find Principal moments.

↳ Principal axes are x, y, z. Symmetry makes I diagonal in this basis.

$$I_x = ma^2 + ma^2 + 2ma^2 = 4ma^2$$

$$I_y = m(2a)^2 + m(2a)^2 + 2m \cdot 0 = 8ma^2.$$

$$I_z = I_x + I_y = 12ma^2.$$

3. Find $\vec{\omega}$ from $\vec{L} = \{I\}\vec{\omega}$ $\quad\vec{L} = (aP, 2aP, 0)$

$$\vec{L} = aP\,\hat{x} + 2aP\,\hat{y} + 0\,\hat{z} = I_x\omega_x\hat{x} + I_y\omega_y\hat{y} + I_z\omega_z\hat{z}$$

$$= 4ma^2\omega_x\hat{x} + 8ma^2\omega_y\hat{y} + 12ma^2\omega_z\hat{z}$$

$$\omega_x = \frac{aP}{4ma^2}, \quad \omega_y = \frac{2aP}{8ma^2}, \quad \omega_z = 0.$$

$$(\omega_x, \omega_y, \omega_z) = \frac{P}{4ma}(1, 1, 0).$$

4. Calculate velocities w.r.t CM.

Right after blow, object rotates about the CM with ang vel found in step 3.

$$\vec{u}_i = \vec{\omega} \times \vec{r}_i$$

$$\vec{u_A} = \vec{\omega} \times \vec{r_A} = \frac{P}{4ma}(1,1,0) \times (-2a,-a,0) = \left(0,0,\frac{P}{4m}\right)$$

$$\vec{u_B} = \vec{\omega} \times \vec{r_B} = \frac{P}{4ma}(1,1,0) \times (2a,-a,0) = \left(0,0,\frac{-3P}{4m}\right)$$

$$\vec{u_C} = \vec{\omega} \times \vec{r_C} = \frac{P}{4ma}(1,1,0) \times (a,a,0) = \left(0,0,\frac{P}{4m}\right)$$

5. Add on vel of CM.

$\vec{P} = (0, 0, -P)$. Total mass $= M = 4m$

Impulse = change in momentum.

$\vec{P} = M\vec{V}_{cm}$, $\vec{V}_{cm} = \frac{\vec{P}}{M} = \left(0, 0, -\frac{P}{4m}\right)$

Total vel. of masses.

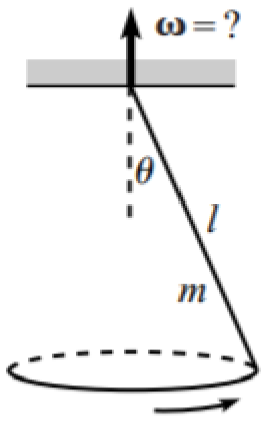
$\vec{v}_A = \vec{u}_A + \vec{V}_{CM}. = (0, 0, 0)$.

$\vec{v}_B = \vec{u}_B + \vec{V}_{CM} = \left(0, 0, \frac{-P}{m}\right)$

$\vec{v}_c = \vec{u}_c + \vec{V}_{CM} = (0, 0, 0)$

Stick of length l, mass m of uniform density.
pivoted at its top end swings around a vertical axis. such that the stick always makes a constant angle θ with vertical.

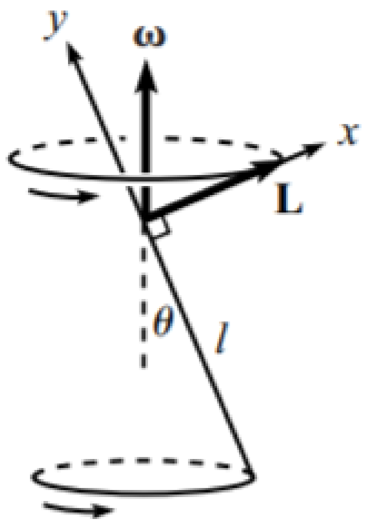
$\vec{\omega} = ?$



$\vec{\omega} = ?$

Strategy

1. Find Principal moments
2. Find $\vec{L}$
3. Find $\frac{d\vec{L}}{dt}$.
4. Use $\vec{\tau} = \frac{d\vec{L}}{dt}$.



- Origin as pivot point.
- Axis along stick + any two axes orthogonal to it as principal axes.
- + z points out of the page.

1. Find Principal moments.

$$I_x = \frac{ml^2}{3}, \quad I_y = 0, \quad I_z = \frac{ml^2}{3}.$$

2. Find $\vec{L}$, $\quad \vec{L} = \{I\}\vec{\omega}$.

$$\vec{\omega} = (\omega \sin\theta, \omega\cos\theta, 0), \quad L_x = I_x\omega_x, \; L_y = I_y\omega_y$$
$$L_z = I_z\omega_z.$$

$$\vec{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z).$$

$$= \left(\frac{1}{3}ml^2\omega\sin\theta, 0, 0\right).$$

<u>Next</u> $\dfrac{d\vec{L}}{dt}$.

as stick rotates, $\vec{L}$ traces out surface of cone.

tip traces out circle of radius $L\cos\theta$.

speed of tip $= (L\cos\theta)\omega$.

$$\left|\frac{d\vec{L}}{dt}\right| = (L\cos\theta)\omega = \frac{1}{3}ml^2\omega^2\sin\theta\cos\theta.$$

points into the page.

Or. alternatively

$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L}$$

$$= (\omega \sin\theta, \omega\cos\theta, 0) \times \left(\frac{1}{3} m l^2 \omega \sin\theta, 0, 0\right)$$

$$= \left(0, 0, -\frac{1}{3} m l^2 \omega^2 \sin\theta \cos\theta\right)$$

Next, calculate torque relative to pivot.

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = rF\sin\theta = \frac{l}{2} m g \sin\theta$$; points into page.

$$\vec{\tau} = \frac{d\vec{L}}{dt} \; ; \quad \frac{\cancel{m} l^{\cancel{2}} \omega^2 \cancel{\sin\theta} \cos\theta}{3} = \frac{\cancel{m} g \cancel{l} \cancel{\sin\theta}}{2}$$

$$\boxed{\omega = \sqrt{\frac{3g}{2l\cos\theta}}}$$