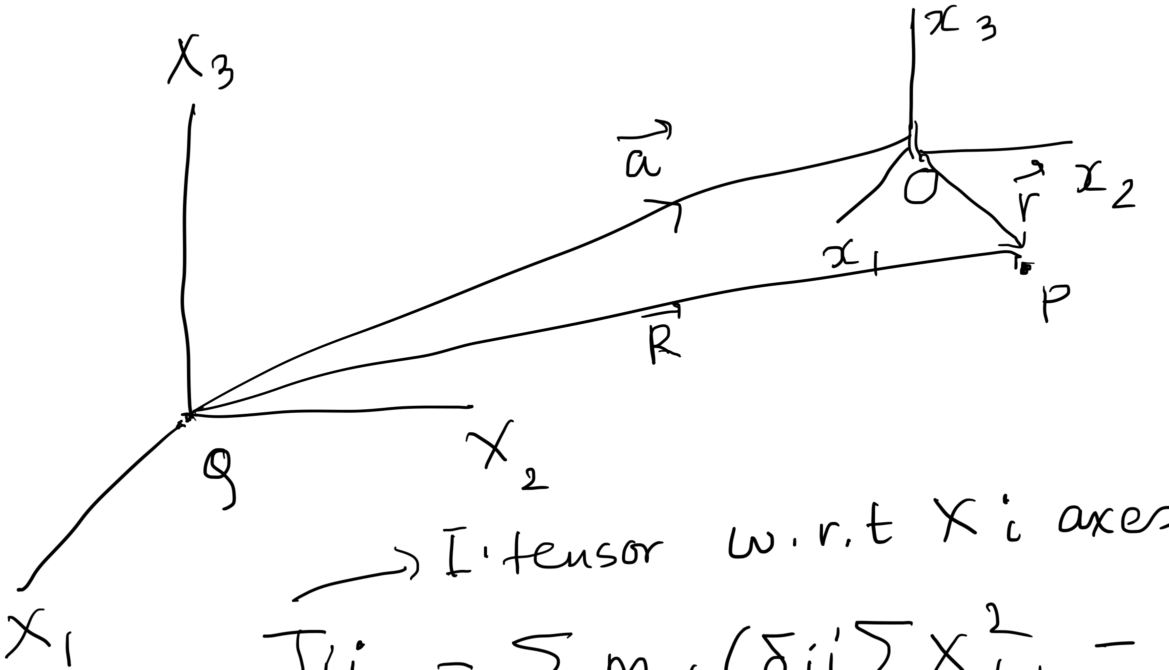
Generalized Parallel-Axis Theorem



→ I tensor w.r.t X_i axes

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j}\right) \quad (1)$$

$$\vec{R} = \vec{a} + \vec{r} \quad (2) \qquad \vec{R} = (X_1, X_2, X_3)$$

$$\vec{r} = (x_1, x_2, x_3)$$

$$X_i = a_i + x_i \quad (3)$$

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k (x_{\alpha,k} + a_k)^2 - (x_{\alpha,i} + a_i)(x_{\alpha,j} + a_j)\right)$$

$$= \sum_\alpha m_\alpha \underbrace{\left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right)}_{I_{ij}}$$

$$+ \sum_\alpha m_\alpha \left\{\left(\delta_{ij} \sum_k (2 x_{\alpha,k} a_k + a_k^2)\right) - (a_i x_{\alpha,j} + a_j x_{\alpha,i} + a_i a_j)\right\} \quad (4)$$

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j\right)$$

$$+ \sum_\alpha m_\alpha \left(2\delta_{ij} \sum_k x_{\alpha,k} a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i}\right)$$

In the last summation each term involves $\sum_\alpha m_\alpha x_{\alpha,k} = 0$, $\sum m_\alpha \vec{r}_\alpha = 0$

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j\right)$$

$$\sum m_\alpha = M, \quad \sum_k a_k^2 = a^2.$$

$$\boxed{I_{ij} = J_{ij} - M\left(a^2 \delta_{ij} - a_i a_j\right)}$$

<u>Th. 1</u> If two principal moments are equal ($I_1 = I_2 = I$) then any axis (through the chosen origin) in the plane of the corresponding principal axis, is <u>also</u> a principal axis, and its moment is also I.

Proof: $\because$ $I_1 = I_2 = I$

if $\vec{u_1}$ and $\vec{u_2}$ are eigenvectors of $\{I\}$

$$\{I\}\vec{u_1} = I\vec{u_1} \quad ; \{I\}\vec{u_2} = I\vec{u_2}$$

$$\{I\}(a\vec{u_1} + b\vec{u_2}) = I(a\vec{u_1} + b\vec{u_2})$$

for all a, b.

$\rightarrow$ any vector in plane spanned by $\vec{u_1}$ & $\vec{u_2}$ is also a soln. $\Longrightarrow$ principal axis

Th.2. If a pancake object is symmetric under a rotation through $\theta \neq 180°$ in the x-y plane, then every axis in the x-y plane (with origin at the centre of symetry rotation) is a principal axis with same moment.

$\vec{\omega_0}$: principal axis in plane

$\vec{\omega_\theta}$; axis obtained by rotating $\vec{\omega}_0$ through θ

$$\therefore \quad \{I\}\vec{\omega_0} = I\vec{\omega_0}$$

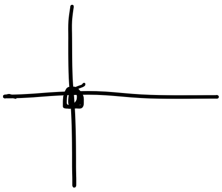
$$\{I\}\vec{\omega_\theta} = I\vec{\omega_\theta}$$

Any vector $\vec{\omega}$ in x-y plane can be written as a linear combination of $\vec{\omega}_0$ and $\vec{\omega}_\theta$, provided that $\theta \neq 180°$ or 0. $\vec{\omega_0}$, $\vec{\omega_\theta}$ span the plane.

$$I\vec{\omega} = \{I\}(a\vec{\omega}_0 + b\vec{\omega}_\theta) = aI\vec{\omega}_0 + bI\vec{\omega}_\theta$$

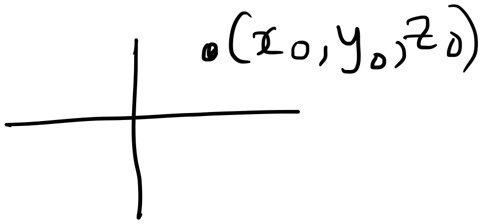
Hence $\vec{\omega}$ is also a principal axis.

Ex:



point mass at origin

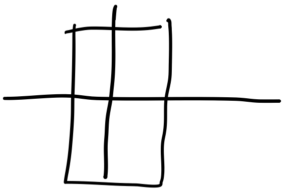
Any axis principal axis



pt mass at (x_0, y_0, z_0).

Axis through the pt.

any axis $\perp$ to it



Rectangle centred at origin

P.A : x, y, z axis

Diagonalized $\{I\}$

$$I_{ij} = I_i \delta_{ij} \quad \text{— (1)}$$

(I_1, I_2, I_3) principal moments of inertia

direction of each principal axis is determined by substituting I_1, I_2, I_3 for I in the eqn.

$$I\omega_1 = I_{11}\omega_1 , \; I\omega_2 = I_{22}\omega_2 , \; I\omega_3 = I_3\omega_3$$

$\hookrightarrow$ determines ratios of ang. vel. vector

Principal axes $\Longrightarrow$ eigenvectors.

$I_1 = I_2 = I_3 \Rightarrow$ spherical top.

$I_1 = I_2 \neq I_3 \Rightarrow$ symmetric top.

$I_1 \neq I_2 \neq I_3 \Rightarrow$ asymmetric top.

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