Physics I

Lecture 6

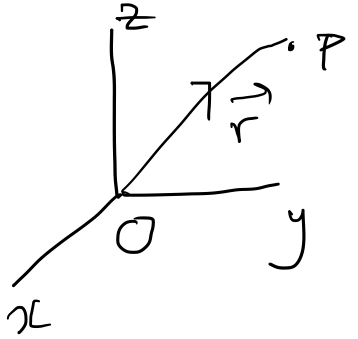
Conservation Laws

1. Linear momentum

$$\dot{\vec{p}} = \vec{F} \quad , \quad \vec{F} = 0 \; , \quad \dot{\vec{p}} = 0$$

$$\boxed{\vec{p} = \text{const}} \; , \text{ when } \vec{F} = 0$$

2. Angular momentum



$$\vec{L} = \vec{r} \times \vec{p}$$

$$\dot{\vec{L}} = \underbrace{\dot{\vec{r}} \times \vec{p}}_{=0} + \underbrace{\vec{r} \times \dot{\vec{p}}}_{\vec{r} \times \vec{F}}$$

$$= \vec{r} \times \vec{F} = \vec{N} \Rightarrow \text{torque}$$

$$\boxed{\vec{L} = \text{const}} \text{ when } \vec{N} = 0$$

3. Work

Work done on a particle by force $\vec{F}$ in taking it from config 1 to config 2.

$$\boxed{W_{12} = \int_1^2 \vec{F}\cdot d\vec{r}}$$

$$\vec{F}\cdot d\vec{r} = m\frac{d\vec{v}}{dt}\cdot d\vec{r} = m\frac{d\vec{v}}{dt}\cdot\frac{d\vec{r}}{dt}\,dt$$

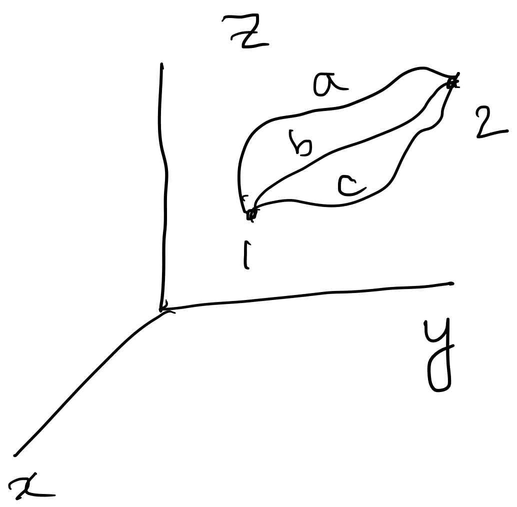
$$= m\frac{d\vec{v}}{dt}\cdot\vec{v} = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v})\,dt$$

$$\vec{F}\cdot d\vec{r} = d\left(\frac{1}{2}mv^2\right).$$

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r}$$

$$= \int_1^2 d\left(\tfrac{1}{2}mv^2\right) = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

$$\boxed{W_{12} = T_2 - T_1} = \Delta T \quad \text{Work-energy theorem}$$



In general

$\int_1^2 \vec{F}\cdot d\vec{r}$ depends on path

But there is a class of forces for which work done does not depend on path. $\Longrightarrow$ Conservative forces.

Conditions for a force to be <u>conservative</u>

(i) $\vec{F}$ depends only on position $\vec{r}$ (not on velocity or time)

$\vec{F} = \vec{F}(\vec{r})$.

(ii) For any two points 1 & 2, $W(1 \rightarrow 2)$ done by $\vec{F}$ must be independent of path.

$\hookrightarrow$ Possible to define a quantity U, called potential energy $U(\vec{r})$

$$\boxed{U(\vec{r}) = -W(\vec{r}_0 \rightarrow \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'}$$

$\vec{r}_0 \rightarrow$ standard position

$$\int_1^2 \vec{F}(\vec{r}\,') \cdot d\vec{r}\,' = \int_1^{\vec{r}_0} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,' + \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

$$= U_1 - U_2 = -\Delta U = -(U_2 - U_1)$$

But recall

$$-(U_2 - U_1) = T_2 - T_1 .$$

$$\boxed{T_1 + U_1 = T_2 + U_2 = E = \text{const}}$$

$\rightarrow$ Total mechanical energy = const for conservative forces.

Non conservative forces

$$\vec{F} = \vec{F}_{cons} + \vec{F}_{NC}.$$

$$\Delta T = W = W_{cons} + W_{nc}.$$

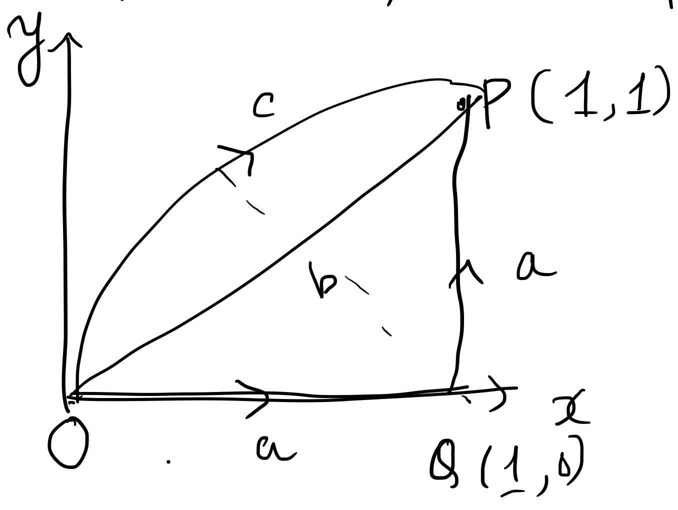
$$= -\Delta U + W_{nc}.$$

$$\boxed{\Delta E = \Delta(T+U) = W_{nc}.}$$

$2 \;,\; \underbrace{y + 2x}_{\text{cannot happen}}$.

1 Digression

example of path dependence of a line integral



$$\vec{F} = y\hat{x} + 2x\hat{y}$$

$$W_a = \int_a \vec{F}\cdot d\vec{r}$$

$$= \int_O^Q \vec{F}\cdot d\vec{r} + \int_Q^P \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 F_x\,dx + \int_0^1 F_y\,dy .$$

$$= \int_0^1 0\,dx + \int_0^1 2\,dy .$$

$$= 2 .$$

$$W_b = \int_b F_x\,dx + \int_b F_y\,dy .$$

$$= 3/2 .$$

Force as a gradient of potential energy

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

suggests that $\vec{F}(\vec{r})$ can be written as some kind of ~~denial~~ derivative of $U(\vec{r})$. In 1d you know

$$F(x) = -\frac{dU}{dx}.$$

$$W(\vec{r} \to \vec{r}+d\vec{r}) = \vec{F}(\vec{r})\cdot d\vec{r} = -dU$$

$$= -\left[U(x+dx,\, y+dy,\, z+dz) - U(x,y,z)\right].$$

In 1d $\quad df = \frac{df}{dx}\,dx.$

$$dU = U(x+dx,\, y+dy,\, z+dz) - U(x,y,z)$$

$$= \left[\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right]$$

$$W(\vec{r} \to \vec{r} + d\vec{r}) = -\left[\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right]$$

$$= F_x dx + F_y dy + F_z dz .$$

$$\Rightarrow \quad F_x = -\frac{\partial U}{\partial x} \; ; \quad F_y = -\frac{\partial U}{\partial y} \; ; \quad F_z = -\frac{\partial U}{\partial z}$$

$$\boxed{\vec{F} = -\hat{x} \frac{\partial U}{\partial x} - \hat{y} \frac{\partial U}{\partial y} - \hat{z} \frac{\partial U}{\partial z}}$$

Gradient :

given any scalar fn $\phi(x,y,z)$.

$$\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

$\hookrightarrow$ vector operator

$$\boxed{\vec{F} = -\vec{\nabla} U}$$

$\vec{\nabla} U \cdot d\vec{r} = dU \Rightarrow$ check.