Physics I

Lecture 7

Conditions for a Force to be Conservative

A force **F** acting on a particle is **conservative** if and only if it satisfies two conditions:

(i) **F** depends only on the particle's position **r** (and not on the velocity **v**, or the time t, or any other variable); that is, $\mathbf{F} = \mathbf{F}(\mathbf{r})$.

(ii) For any two points 1 and 2, the work $W(1 \rightarrow 2)$ done by **F** is the same for all paths between 1 and 2.

$$U(\vec{r}) = -W(\vec{r}_0 \rightarrow \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

Total Mechanical energy

$$E = T + U \quad \text{conserved}$$

$$\vec{F} = -\vec{\nabla} U$$

"Any conservative force is derivable from a potential energy"

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

vector operator

$$\vec{\nabla} U = \hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}$$

$$\vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$$

$$F_x = -\frac{\partial U}{\partial x} \quad ; \quad F_y = -\frac{\partial U}{\partial y} \quad ; \quad F_z = -\frac{\partial U}{\partial z}$$

example: $U = Axy^2 + B\sin Cz$ A, B, C are constants

$$\vec{F} = -\vec{\nabla} U$$

$$F_x = -\frac{\partial U}{\partial x} = -Ay^2$$

$$F_y = -\frac{\partial U}{\partial y} = -2Axy$$

$$F_z = -\frac{\partial U}{\partial z} = -CB\cos Cz$$

2^{nd} Condition for $\vec{F}$ to be conservative

Recall for $\vec{F}$ to be conservative

$$W(\vec{r}_0 - \vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}' \longrightarrow \text{path independent}$$

Can we find a differential equivalent criterion to test whether a force is conservative?

Yes.

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

curl of a vector, Given $\vec{A}$, $\boxed{\text{curl}\vec{A} = \vec{\nabla} \times \vec{A}}$

It can be shown that $\int_1^2 \vec{F}\cdot d\vec{r}$ is independent of path iff

$$\boxed{\vec{\nabla} \times \vec{F} = 0}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)$$

$$+ \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

If $\vec{F} = -\vec{\nabla} U$, $\vec{\nabla}\times\vec{F} = 0$

[Ex : $\vec{\nabla}\times\vec{\nabla}\phi = 0$ show : identity].

Coulomb Force

$$\vec{F} = \frac{kqQ}{r^2}\hat{r} = \frac{\alpha}{r^3}\vec{r} = \frac{\alpha}{r^3}(x\hat{x} + y\hat{y} + z\hat{z}).$$

$$\vec{\nabla} \times \vec{F} = 0 \ ?$$

$$F_x = \frac{\alpha x}{r^3}, \quad F_y = \frac{\alpha y}{r^3}, \quad F_z = \frac{\alpha z}{r^3}.$$

$$(\vec{\nabla} \times \vec{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}.$$

$$= \frac{\partial}{\partial y}\left(\frac{\alpha z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{\alpha y}{r^3}\right).$$

$$= -(3\alpha yz)/r^5 + \frac{3\alpha yz}{r^5} = 0$$

$$(\vec{\nabla}\times\vec{F})_x = (\vec{\nabla}\times\vec{F})_y = (\vec{\nabla}\times\vec{F})_z$$

$$\boxed{\vec{\nabla}\times\vec{F} = 0} \quad \vec{F} \rightarrow \text{conservative}.$$

Potential energy exists.

$$U(\vec{r}) = \frac{\alpha}{r} \qquad r_0 \rightarrow \infty \qquad U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r})\cdot d\vec{r}$$

$$U(\vec{r}) = \frac{kqQ}{r} \qquad \vec{F} = -\vec{\nabla}U.$$

$$(\vec{\nabla}U)_x = \frac{\partial}{\partial x}\left(\frac{kQq}{r}\right) = \frac{-kqQ}{r^2}\frac{\partial r}{\partial x} = \frac{-kQqx}{r^3}$$

$$\left\{ \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \right\}$$

$$(\vec{\nabla} U)_y = \frac{-kqQy}{r^3} = F_y \quad \text{and so on}$$

$$\vec{F} = -\vec{\nabla} U = \frac{kqQ\vec{r}}{r^3} = \frac{kQq}{r^2}\hat{r}.$$

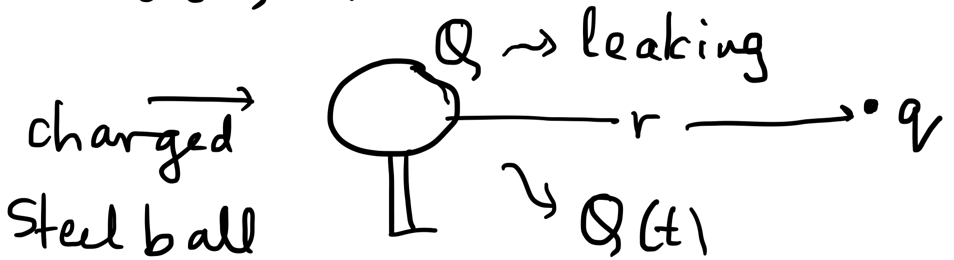
<u>Time dependent potential energy</u>

$\vec{F}(\vec{r},t)$ say $\vec{F}$ satisfies $\vec{\nabla} \times \vec{F} = 0$

but not 1st condition.

non-conservative, but can still define

$U(\vec{r},t)$ $\qquad \vec{F} = -\vec{\nabla} U$



$$\vec{F} = \frac{k\,Q(t)\,q}{r^2}\hat{r}$$

$$\vec{F} = \frac{k q Q(t)}{r^2} \hat{r}$$

$$\vec{\nabla} \times \vec{F} = 0$$

$$\vec{F} = -\vec{\nabla} U(\vec{r}, t)$$

$$E = T + U \quad ; \quad dT = \frac{dT}{dt} dt = (m \dot{\vec{v}} \cdot \vec{v}) dt$$
$$= \vec{F} \cdot d\vec{r}$$

$$dU = \underbrace{\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz} + \frac{\partial U}{\partial t} dt .$$

$$dU = \vec{\nabla} U \cdot d\vec{r} + \left(\frac{\partial U}{\partial t}\right) dt$$

$$dU = \vec{\nabla} U \cdot d\vec{r} + \frac{\partial U}{\partial t}\, dt$$

$$= -\vec{F} \cdot d\vec{r} + \frac{\partial U}{\partial t}\, dt$$

$$dE = d(T + U)$$

$$= dT + dU$$

$$= \cancel{\vec{F} \cdot d\vec{r}} - \cancel{\vec{F} \cdot d\vec{r}} + \frac{\partial U}{\partial t}\, dt$$

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} = 0 \quad \text{only when } \frac{\partial U}{\partial t} = 0$$

$E = U + T$ conserved

U does not explicitly depend on time