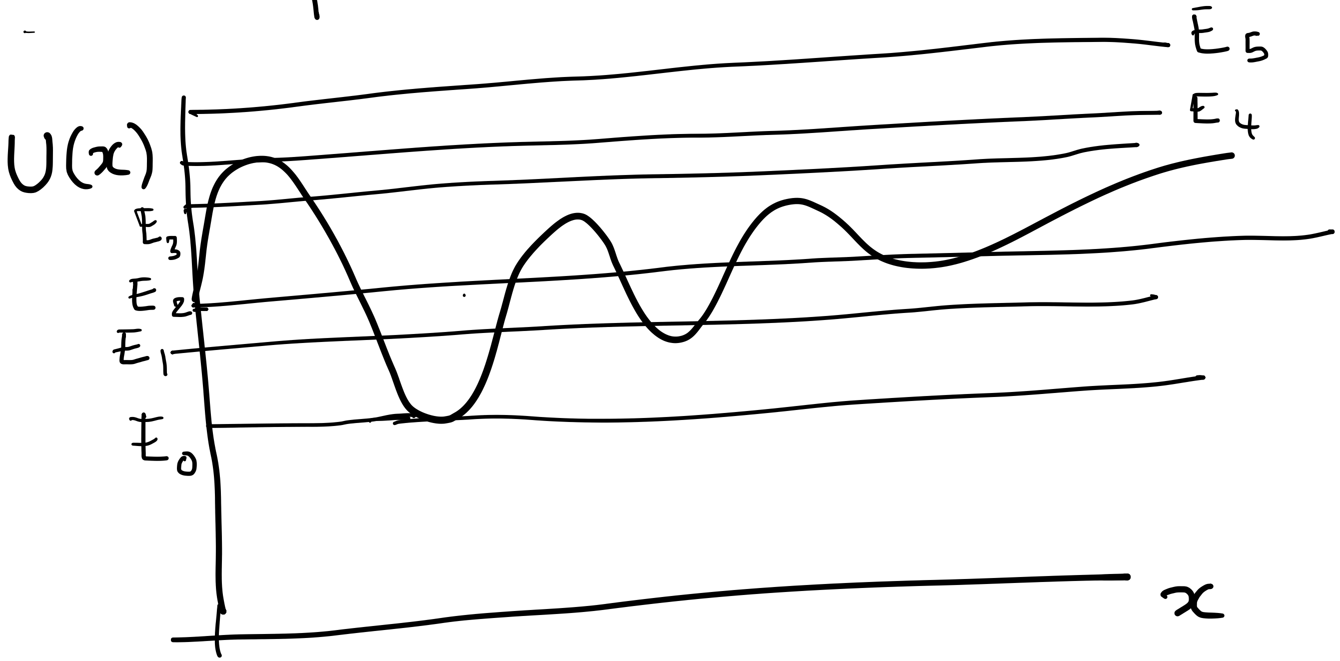
Physics I

Lecture 9

Recap: 1 d linear motion



- Bounded vs unbounded
- turning points
- classically allowed regions. $U(x) \leq E$
- stable vs unstable equilibrium

Completely solvable system

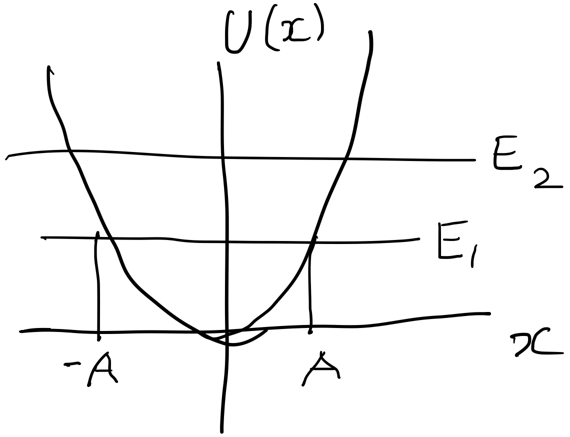
$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

can be formally integrated $\longrightarrow$ $t - t_0 = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \left[E - U(x')\right]^{-1/2} dx'$

Oscillations

Simple harmonic oscillator

$$F = -kx \ , \quad U(x) = \frac{1}{2} k x^2$$



$\longrightarrow$ for all E motion is bounded and oscillatory

Turning points are symmetric about the origin

Equation of motion
$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

S.H.O is an exactly solvable problem

Any potential can be approximated as S.H.M close to a minimum of potential

$$U(x) \simeq \underset{\substack{\shortparallel \\ \text{can be} \\ \text{set to zero}}}{U(x_0)} + \underbrace{\left.\frac{dU}{dx}\right|_{x=x_0}}_{=0} (x-x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0} (x-x_0)^2 + \cdots$$

$$\simeq \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0} (x-x_0)^2 .$$

Linear ordinary differential eqns with constant coefficients

no higher than 1st degree in dependent variable and derivatives.

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1(t)\frac{dx}{dt} + a_0(t) = b(t).$$

order n

$b(t) = 0 \Rightarrow$ homogeneous

We will mostly be concerned with 2nd order.

General soln of any 2nd order eqn will contain 2 arbitrary constants

$$x = x(t; c_1, c_2)$$

Theorem 1 : If $x = x(t)$ is a soln of any a linear homogeneous differential eqn, then $x_1 = C x_0(t)$ is also a soln where C is a const.

Theorem 2 : If $x = x_1(t)$ and $x = x_2(t)$ are solutions of a linear homogeneous differential eqn, then $\boxed{x = x_1(t) + x_2(t)}$ is also a soln.

2nd order.

General soln is given by $C_1 x_1(t) + C_2 x_2(t)$ where x_1 and x_2 are linearly independent solns and C_1 and C_2 are real constants.

$x_1(t)$ and $x_2(t)$ are said to be linearly independent iff

$$\lambda x_1(t) + \mu x_2(t) = 0 \quad \text{only for } \begin{array}{l}\lambda = 0\\ \mu = 0\end{array}$$

Consider 2nd order diff eqns with const coeff.

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \quad \text{— (1)}$$

Assume a soln of form $x = e^{pt}$, {Ansatz}.

plug ansatz into (1)

$$\boxed{a_2 p^2 + a_1 p + a_0 = 0} \quad \text{— (2)} \implies \text{auxilliary eqn}$$

$$p^2 + ap + b = 0 \qquad a = \frac{a_1}{a_2}, \; b = \frac{a_0}{a_2}$$

$$p = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

p_1, p_2 ; If $p_1 = p_2$, 1 soln.

for $p_1 \neq p_2$

$$\boxed{x = c_1 e^{p_1 t} + c_2 e^{p_2 t}}$$

If $p = p_1 = p_2$, only one soln. by this method, Verify that te^{pt} is also a soln. and is linearly independent.

Harmonic Oscillator

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

$x \sim e^{pt} \longrightarrow p^2 + \omega_0^2 = 0$ auxillary eqn.

$$p = \pm i\omega_0$$

$x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$, In general C_1, C_2 can be complex

but x must be real so must restrict C_1, C_2 accordingly

$C = C_2 = C_1^*$ will ensure a real soln.

$$x = C e^{i\omega_0 t} + C^* e^{-i\omega_0 t}. \qquad C = \frac{1}{2} A e^{-i\delta}$$

$$\boxed{x = \underbrace{A}\cos(\omega_0 t - \underbrace{\delta}).}$$

2 arbitrary constants.

equivalently written

$$x = B_1 \cos\omega_0 t + B_2 \sin\omega_0 t$$

or

$$x = A \cancel{\cos} \sin(\omega_0 t - \phi)$$

Another comment

$$\ddot{x} + \omega_0^2 x = 0$$

↳ since contains only real coeff → soln. will be real.

A complex fn. can satisfy this diff, its real and imaginary parts satisfy ~~to~~ it separately.

say $x = C e^{i\omega_0 t} = A e^{i(\omega_0 t - \delta)} = A\cos(\omega_0 t - \delta) + i A \sin(\omega_0 t - \delta)$.