B.Math. (Hons.) IInd year
Second Semester 2022-2023
Rings and Modules ::: B.Sury
Assignment 0 - Not for Submission
These are for practice; let me know if you can't solve a problem.

General conventions:

- Rings R are possibly without unity.
- A sub-ring S of a ring R is a non-empty subset which is a subgroup of $(R, +)$ and is closed under the product map.
- The center $Z(R)$ of a ring is the sub-ring of all those elements which commute with every element.
- An element e in a ring is called *idempotent* if $e^2 = e$.
- A ring is said to be Boolean if every element is idempotent.
- In a ring R, an element $x \in R$ is said to be *nilpotent* if $x^n = 0$ for some $n \geq 1$.
- An element in a ring with unity is said to be a unit if it has a multiplicative inverse.
- A ring R with unity in which $R \setminus \{0\}$ is a group under multiplication, is called a *field* (resp. *division ring*) if R is commutative (resp. non-commutative).

Q 1. If R is a ring in which $x^2 - x$ belongs to the center $Z(R)$ for each $x \in R$. Prove that R must be commutative.

Q 2. Let R be a ring in which $x^2 = 0$ implies $x = 0$. Show that any idempotent e of R must be in the center $Z(R)$. In particular, in a ring R in which the only nilpotent element is 0, each idempotent element is contained in $Z(R)$.

Q 3. Let R be a commutative ring with unity. If u is unit, and a is nilpotent, prove that $u + a$ is a unit.

Q 4. Give an example of a ring and nilpotent elements a, b in it such that $a + b$ is not nilpotent. Can this happen in a commutative ring? Why or why not?

Q 5. Let d be a square-free positive integer. Consider the rings $\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$ and $\mathbb{Z}[\sqrt{-d}] := \{a + b\sqrt{-d} : a, b \in \mathbb{Z}\}$ under the usual addition and multiplication of complex numbers. Show that the first ring has infinitely many units. Further, find all the units in the second ring.

Q 6. Let R be any ring in which the equation $ax = b$ has solutions for any $a \neq 0$ and $b \in R$. Prove:
(i) R has no (left or right) zero-divisors other than 0,
(ii) R has a unity,
(iii) R is a division ring or a field.

Q 7. Let R be a commutative ring and $f \in R[X]$ be a polynomial such that $fg = 0$ in $R[X]$ for some polynomial $g \neq 0$. Show that there exists $r \neq 0$ in R such that $rf = 0$.

Q 8. Prove that for a commutative ring R, a polynomial $f = c_0 + c_1 X + \cdots + c_n X^n$ in $R[X]$ is nilpotent (as an element of $R[X]$) if, and only if, each $c_i \in R$ is nilpotent. Further, show that a polynomial $g = a_0 + a_1 X + \cdots + a_d X^d \in R[X]$ is a unit in $R[X]$ if and only if, $a_0 \in R$ is a unit and $a_1, \cdots, a_d \in R$ are nilpotent.

Q 9. Let D be a division ring. Suppose the center $Z(D)$ is an infinite field. Then, show that each element $a \in D^*$ which has only finitely many conjugates under D^*, must lie in $Z(D)$.
Hint: Show first that the centralizer

$$C(a) := \{x \in D^* : xa = ax\}$$

has finite index in the group D^*. If the index is not 1, let $x \in D^* \setminus C(a)$. Then, look at the set of conjugates $(c+x)a(c+x)^{-1}$ as c varies in $C(a)$ and deduce a contradiction.
(The above exercise generalizes "Wedderburn's little theorem" which states that a finite division ring must be commutative.)

Q 10.
(a) If R is a ring in which $x^3 = x$ for all $x \in R$, then prove that R must be commutative.
(b) If R is a ring in which $x^4 = x$ for all $x \in R$, then prove that R must be commutative.
(c) Let R be a ring with unity 1. If $(xy)^r = xy$ is satisfied for three consecutive natural numbers r, and for all $x, y \in R$, prove that R must be commutative.