Assignment I - Rings and Modules
B. Math. Hons. IInd year
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Q 1. Let R be a Boolean ring (that is, each element is idempotent). If $a_1, \cdots, a_n \in R$, then show that the ideal $(a_1, \cdots, a_n)$ generated by $a_1, \cdots, a_n$ is principal.

Q 2. Let G be a non-trivial finite group. Let R be any commutative ring with unity. Prove that the group ring $R[G]$ has nontrivial zero divisors.

Q 3. Let m, n be positive integers > 1. Describe all the ring homomorphisms from $\mathbb{Z}_m$ to $\mathbb{Z}_n$ (note that by a homomorphism we mean a map that respects respective addition and multiplication operations but may not take 1 to 1). How many are they in number? Is it the same when m and n are interchanged?

Q 4. Recall we defined an element p of a commutative ring to be irreducible if $p = ab$ for some $a, b \in R$ implies either a or b is a unit. Consider the reducible polynomial $X^2 + 3X + 2$ in $\mathbb{Z}[X]$. Find a bigger ring (which is not a field) in which it is irreducible.

Q 5. Let $f_1, f_2, \cdots, f_r$ be polynomials in $\mathbb{R}[X_1, \cdots, X_n]$. Prove that the set

$$V(f_1, \cdots, f_r) := \{(a_1, \cdots, a_n) \in \mathbb{R}^n : f_i(a_1, \cdots, a_n) = 0 \ \forall \ i \leq r\}$$

equals

$$\{(b_1, \cdots, b_n) \in \mathbb{R}^n : f(b_1, \cdots, b_n) = 0\}$$

for some polynomial $f \in \mathbb{R}[X_1, \cdots, X_n]$.
Prove the analogous result with $\mathbb{R}$ replaced by any field F which is not algebraically closed (the last phrase means there are non-constant polynomials in $F[X]$ which do not have any roots in F).

Q 6. Recall that the real vector space H of Hamilton's real quaternions has a basis $1, i, j, k$ and it has a ring structure by declaring $i^2 = j^2 = k^2 = -1, ij = -ji = k$ and associativity. Find a subring of $M_2(\mathbb{C})$ isomorphic to H (isomorphic means there is a bijection that preserves the respective addition and multiplication operations). Write an explicit isomorphism.

Q 7. In the ring $C([0,1], \mathbb{R})$, find an infinite sequence of ideals

$$I_1 \subset I_2 \subset I_3 \subset I_4 \subset \cdots\cdots$$

that is strictly increasing (that is, it does not stop).

Q 8. For any positive integer n, find a commutative ring with unity which has exactly n maximal ideals.