Assignment II - Rings and Modules
B. Math. Hons. IInd year
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Unless specified other wise, all rings mentioned below are commutative and have unity.

Q 1. (Parts of Ex. 1.18 on page 10 of A-M)
If $f : A \to B$ is a ring homomorphism and I_1, I_2 are ideals in A and J_1, J_2 are ideals in B, give examples to show that the following containments can be strict:

$$(I_1 \cap I_2)^e \subset I_1^e \cap I_2^e; (I_1 : I_2)^e \subset I_1^e : I_2^e;$$

$$J_1^c + J_2^c \subset (J_1 + J_2)^c; (J_1 : J_2)^c \subset J_1^c : J_2^c.$$

Q 2. (Ex. 18, Chapter 1 of Atiyah-Macdonald)
For a ring A, the set X of all prime ideals of A is called the (prime) spectrum of A. It is a topological space where the closed sets are, by definition, the sets of the form $V(E) := \{$ all prime ideals of A containing $E\}$ for any subset E of A. For instance $V(\{0\}) = X$ and $V(\{1\}) = \emptyset$. Assume the properties mentioned in exercises 15 and 17 which describe this topological space; this is called the Zariski topology on X.
For psychological reasons, it is convenient to think of a point of X as x, but think of it as an ideal P_x, since x is really a prime ideal of A. Prove:
(a) $V(P_x) = \overline{\{x\}}$;
(b) $P_x \subseteq P_y$ iff $y \in \overline{\{x\}}$;
(c) X is a T_0-space; that is, given any two distinct points in X, at least one of them has an open neighbourhood not containing the other point.
Hint for (c). If $x \neq y$ in X, either P_x is not contained in P_y or P_y is not contained in P_x. In the first case, consider the open set $\left(\overline{\{x\}}\right)^c$.

Q 3.
(a) Let (A, δ) be a Euclidean domain; that is, δ is a Euclidean function on the domain A. Then, prove that the quotient and remainder are unique if and only if $\delta(a + b) \leq max(\delta(a), \delta(b))$ for all $a, b, a + b \neq 0$.
(b) Let $d : A \setminus \{0\} \to \{0, 1, \cdots\}$ be a function (where A is a domain) satisfying only the second property of a Euclidean function; viz., for each $a, b \neq 0$, there exist $q, r \in A$ with $a = qb + r$ and either $r = 0$ or $d(r) < d(b)$. Show

that (A, δ) is a Euclidean domain, where $\delta(a) := min\{d(ab) : b \neq 0\}$. In other words, δ satisfies both the properties of a Euclidean function.
(c) In $\mathbf{Z}[\sqrt{-7}]$, for each $k \geq 2$, show that there is an element which is a product of $2k, 2k+1, \cdots, 3k$ irreducible elements at the same time.
Remark. The exercise (b) refers to a question one of you asked in class as to why the first property was needed; it is not needed but it is forced as noted in (b) and it is useful in factorizing elements into irreducibles.

Q 4. Show that a prime p is congruent to 1 or 3 mod 8 if, and only if, it is expressible as $x^2 + 2y^2$.
Hint. You may assume that -2 is a square mod p for such primes. Then, use the fact that $\mathbb{Z}[\sqrt{-2}]$ is a ED (and hence a UFD).

Q 5.
(a) Prove that $\mathbb{R}[X,Y]/(X^2+Y^2-1)$ is not a UFD.
(b) Prove that $\mathbb{C}[X,Y]/(X^2+Y^2-1)$ is a PID (and so, a UFD).
Remark: Unlike (a) and (b) above, the ring $\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1)$ *is a UFD whereas* $\mathbb{C}[X,Y,Z]/(X^2+Y^2+Z^2-1)$ *is not a UFD!*

Q 6. Prove that the following are NOT UFDs:
$\mathbb{Z}[2i], \mathbb{Z}[\sqrt{8}], \mathbb{Z} + X\mathbb{Q}[X]$.