Rings and Modules
Assignment III
Due on 27th March 2023

Throughout below, R denotes a commutative ring with unity and, by an R-module, we mean a left R-module which is unital; that is, $1.m = m$.

Q 1.
(i) Consider a finitely generated R-module M. If $M_1 \subseteq M_2 \subseteq M$ be submodules such that $M/M_1 \cong M/M_2$ as R-modules, prove that $M_1 = M_2$. If M is not finitely generated, then is the above necessarily true?
(ii) If $N_1 \cong N_2$ as R-modules, is it necessarily true that the annihilator ideals are the same; that is, $ann_R(M) = ann_R(N)$?

Q 2.
(i) Let M be a finitely generated R-module, and let $\phi : M \to M$ be a surjective R-module homomorphism. Prove that ϕ must be an isomorphism.
(ii) Recall that R is said to be Noetherian if all ideals are finitely generated. Equivalently, every ascending sequence of ideals $I_1 \subseteq I_2 \subseteq \cdots\cdots$ stabilizes; that is, there exists $n_0 \geq 1$ such that $I_n = I_{n_0}$ for all $n > n_0$. If R is Noetherian, and $\theta : R \to R$ is a surjective ring homomorphism, prove that θ must be an isomorphism.
Hint. Get an increasing sequence of ideals with the aid of θ.

Q 3.
(i) If M is a finitely generated R-module where R is Noetherian, prove that all submodules of M are necessarily finitely generated.
(ii) If $N_1 \subseteq N_2$ are R-modules such that N_1 and N_2/N_1 are finitely generated as R-modules, then N_2 must be finitely generated as well.

Q 4. Prove that every R-module is free if, and only of, R is a field.

Q 5. (Generalization of a theorem of Cohen due to Jothilingam)
If M is a finitely generated R-module. If PM is a finitely generated submodule of M for each prime ideal P of R, show that all submodules of M are finitely generated as well.
Hint: If N is maximal among non-finitely generated submodules of M, the annihilator ideal $ann_R(M/N)$ is prime.

Q 6. Let

$$0 \to M' \overset{\alpha}{\to} M \overset{\beta}{\to} M" \to 0$$

be a short exact sequence of R-modules which splits. Recall this means α is 1-1, β is onto, $Ker(\beta) = Image(\alpha)$, and there exists a 'splitting' (an R-module homomorphism) $s : M" \to M$ such that $\beta \circ s = Id_{M"}$. Prove that the set of all splittings $s : M" \to M$ is in bijection with the set $\mathrm{Hom}_R(M", M')$ of all R-module homomorphisms from $M"$ to M'.

Q 7. (Local-Global principle)
For a multiplicative subset S of R and an R-module M, define a relation on the set $M \times S$ by

$$(m_1, s_1) \sim (m_2, s_2)$$

if, and only if, there exists $s \in S$ such that $s(s_1 m_2 - s_2 m_1) = 0$. This is an equivalence relation (assume this - you can verify it but not show the calculation here). Denoting the equivalence class of $(m.s)$ by $\frac{m}{s}$, the equivalence classes form an $S^{-1}R$-module under the addition

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_1 m_2 + s_2 m_1}{s_1 s_2}$$

and the scalar multiplication

$$\frac{r}{s} . \frac{m}{t} = \frac{rm}{st}.$$

Assume this also - verify it for yourself or look up the routine proof.
Prove that if $S^{-1}M$ is the zero $S^{-1}R$-module with $S = R - \mathfrak{m}$ for each maximal ideal $\mathfrak{m}$, prove that $M = (0)$.
Hint: If $0 \neq x \in M$, consider a maximal ideal $\mathfrak{m}$ containing $ann_R(M)$, and show $S^{-1}(M) \neq (0)$ where $S = R - \mathfrak{m}_0$.