Rings and Modules
Assignment IV
Due on 12th April 2023

Q 1.
(a) Discuss whether the abelian group $\mathbb{Z}/m\mathbb{Z}$ can be written as the direct sum of two proper sub groups, where $m = p^2q^3r^4$ are p, q, r are distinct primes.
(b) Determine the number of non-isomorphic abelian groups of order 360.

Q 2.
(a) Find a base for the submodule M of $\mathbb{Z}^3$ generated by the elements $(1, 0, -1), (2, -3, 1), (0, 3, 1), (3, 1, 5)$.
(b) Let R be a PID. Prove that a vector $(a_1, a_2, \cdots, a_n)$ in R^n can be completed to a basis if, and only if, the ideal $(a_1, a_2, \cdots, a_n) = (1)$.

Q 3.
(a) Find the invariant factors (that is, the Smith normal form) of

$$\begin{pmatrix} X-17 & 8 & 12 & -14 \\ -46 & X+22 & 35 & -41 \\ 2 & -1 & X-4 & 4 \\ -4 & 2 & 2 & X-3 \end{pmatrix}.$$

(b) Find all possible Jordan forms of a matrix whose characteristic polynomial is $(X+2)^2(X-5)^3$.

Q 5. Let $A \in M_n(\mathbb{Z})$ and consider the group homomorphism T_A from $\mathbb{Z}^n$ to itself given by $v \mapsto Av$ (where v is written as a column). Find necessary and sufficient conditions for the image of T_A to have finite index in $\mathbb{Z}^n$. When that condition holds, determine the index.

Q 6. Let $A \subset B \subset C$ be commutative rings. If C is finitely generated as a B-module and B is finitely generated as an A-module, then prove that C is finitely generated as an A-module.

Q 7. Let k be a field. Prove that two matrices $A, B \in M_n(k)$ are similar if, and only if, $XI - A$ and $XI - B$ have the same invariant factors as elements of $M_n(k[X])$.

Q 8. *In this problem, comments about fundamental groups are made for interest; they may safely be ignored and the relevant problem on the structure of the finitely generated abelian group can be solved.*
The Klein bottle is a 'surface' whose fundamental group G has a presentation $< a, b|ab = b^{-1}a >$. Show that $G/[G, G] \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
The braid group B_n for $n \geq 3$ is the group with a presentation

$$< g_1, g_2, \cdots, g_{n-1}|g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, g_ig_j = g_jg_i \; if \; |i-j| \geq 2 > .$$

That is, the latter relations hold only for i, j such that $|i - j| > 1$.
(It is an interesting fact that the fundamental group of the complement of the trefoil knot (see figure below) is B_3. Knots are embeddings of S^1 in S^3 and are distinguished usually by the fundamental group of their complements).
Show that the abelianization $B_n/[B_n, B_n]$ of B_n is isomorphic to $\mathbb{Z}$.
For both these problems, the hint is to view any relation $x_1^{i_1} \cdots x_r^{i_r} = 1$ in a group G additively as $i_1x_1 + \cdots + i_rx_r = 0$ in $G/[G, G]$.

A poem by Leo Moser is:
A mathematician named Klein
thought the Möbius band was divine.
Said he: "If you glue
The edges of two,
You'll get a weird bottle like mine."