Some miscellaneous problems on modules over PIDs etc.
For B Math. Hons. 2nd year
Rings and Modules 2023
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Q 1. If an $n \times n$ matrix is in block form $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, prove that $det(A) = det(B)det(D)$.

Q 2. Give an example of a ring R with identity such that $R^2 \cong R$ as R-modules.

Q 3. Find an expression for the number of abelian groups of order n.

Q 4. Find the rational canonical forms of the matrices $\begin{pmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 4 & -3 \end{pmatrix}$, and the Jordan form of the matrix $\begin{pmatrix} -1 & 1 & 1 \\ -3 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$.

Q 5. Find all the matrices of order 3 up to conjugacy (that is, similarity) in $GL_5(\mathbb{Q})$.

Q 6. Formulate a Jordan real canonical form.
Hint: The only irreducible polynomials of degree > 1 over $\mathbb{R}$ are $(x-u)^2+v^2$ for real u, v with $v \neq 0$.

Q 7. If $L \supset K$ are fields, and $A, B \in M_n(K)$ are conjugate in $GL_n(L)$, then show that they are conjugate in $GL_n(K)$ also. Using this and the Jordan form, prove directly that over any field, a matrix and its transpose are conjugate.

Q 8. Let $A \leq \mathbb{Z}^n$ be the abelian group generated by the elements $f_i = \sum_{j=1}^{n} a_{ij} e_j$, where the matrix $(a_{ij}) \in M_n(\mathbb{Z})$ has non-zero determinant. Prove that $\mathbb{Z}/A$ is finite, and find its order.

Q 9. (Did this in class but you may re-do it) Let K be a field, and $A, B \in M_n(K)$. If $XI_n - A$ and $XI_n - B$ are equivalent in $M_n(K[X])$ (that is, $P(XI_n - A)Q = XI_n - B$ for some $P, Q \in GL_n(K[X])$), prove that

A and B are conjugate by a matrix in $GL_n(K)$.
Note that $GL_n(K[X])$ consists of matrices with entries from $K[X]$ whose determinant is in $K[X]^* = K^*$.

Q 10. Prove the following theorem due to Frobenius.
If $A \in M_n(K)$ with K, a field, and if $f_1(X), f_2(X), \cdots, f_r(X)$ are the invariant factors of $XI_n - A$ (different from 1), prove that the matrices in $M_n(K)$ commuting with A forma vector space of dimension $\sum_{i=1}^{r}(2r-2i+1)deg(f_i)$.

Q 11. Let G be the abelian group $\oplus_{i=1}^{r}\mathbb{Z}/d_i\mathbb{Z}$ where $d_1|d_2|\cdots|d_r$. Prove that the number of endomorphisms of G (group homomorphisms of G to itself) equals $\prod_{i=1}^{r}|d_i|^{2r-2i+1}$.
If G above is not cyclic, can the group $Aut(G)$ be abelian?

Q 12. Let R be a PID and $a \in R$. If $A \in M_n(R)$ has determinant in $1+aR$, show there exists a matrix $B \equiv A$ mod aR such that $det(B) = 1$. Here, $B \equiv A$ mod aR means that each entry $b_{ij} - a_{ij} \in aR$.

Q 13. Consider a linear differential equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ where a_i's are real numbers. Introduce new variables $x_1 = y, x_2 = x_1', \cdots, x_n = x_{n-1}'$. Then the DE is equivalent to the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n. \end{aligned}$$

Discuss how to solve this system by writing it in the form

$$x' = C(f)x$$

where $C(f)$ is the companion matrix of the polynomial $f = X^n + a_{n-1}X^{n-1} + \cdots + a_0$.
As an example, find all solutions of

$$y^{(3)} - y^{(2)} + 4y' - 4y = 0.$$

Q 14. If $A \in M_n(\mathbb{C})$ has trace 0, prove that there exist $B, C \in M_n(\mathbb{C})$ such that $A = BC - CB$.

Is this true for matrices over $\mathbb{Z}$?

Q 15. Let $A, B \in M_n(\mathbb{C})$ have disjoint spectra. Let $f, g \in \mathbb{C}[X]$ be arbitrary. Prove that there exists $h \in \mathbb{C}[X]$ such that $h(A) = f(A), h(B) = g(B)$.

Q 16. Prove the Jordan decomposition:
If $A \in M_n(\mathbb{C}$, then there exist matrices S, N which are polynomial expressions in N without constant term such that S is 'semisimple' (that is, the minimal polynomial has distinct roots) and N is nilpotent. Further, show that if $A = S_1 + N_1$ for some commuting matrices S_1, N_1 which are semisimple and nilpotent respectively, then $S = S_1$ and $N = N_1$.

Q 17. If $A \in M_n(\mathbb{R})$ has all eigenvalues real and positive, prove that for each $k \geq 1$, A has a k-th root in $M_n(\mathbb{R})$.

Q 18. Let V be a real vector space of dimension $n \geq 3$. If $T \in End(V)$ (that is, it is a linear transformation), then prove that there exists a subspace W of dimension 1 or 2 such that $T(W) \subset W$.
Hint: If not, then V is an irreducible as an $\mathbb{R}[X]$-module, where $X.v = T(v)$. What do finitely generated $\mathbb{R}[X]$-module look like?

Q 19. If $A \in M_n(\mathbb{C})$ does not have n distinct eigenvalues, prove that $AN = NA$ for some nilpotent matrix $N \neq 0$.

Q 20. Let V be a $\mathbb{Q}$-vector space of dimension n. Suppose $T \in GL(V)$ (invertible linear transformation) satisfying $T^{-1} = T + T^2$, prove that $3|n$.

Q 21. Let V be a K-vector space of finite dimension over a field K. Let $W \subseteq V$ be a subspace of V and $T \in End(V)$ such that $T(W) \subseteq W$. So, T can be regarded as linear transformations of W as well as of V/W. If $m_V, m_W, m_{V/W}$ denote the minimal polynomials of T over $V, W, V/W$ respectively, then prove that m_V divides $m_W m_{V/W}$.
Further, give an example to show that m_V may not be equal to $m_W m_{V/W}$.