Mock Exam - Open Book
B. Math. (Hons.) IInd year
Algebra II
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Q 1. Show that $\mathbf{Z}[X]$ is not a Euclidean domain.

Q 2. Let M be a free module of finite rank over a PID. Show that any submodule $N \neq 0$ of M is free, of rank $r \leq rank(M)$.

Q 3. Use the fact that $\mathbf{Z}[i]$ is a UFD to determine all solutions of $x^2+1 = y^3$ in integers x, y.

Q 4. Given a matrix $M \in M_n(K)$, where K is a field, what is meant by its rational canonical form? Further, by assuming the existence of the rational canonical form, compute the characteristic polynomial of M.

Q 5. Prove that the rings $\mathbb{C}[X]/(X^2+1)$ and $\mathbb{C}[t, t^{-1}]$ are isomorphic.

Q 6. Prove that the norm $N(a + b\omega) = a^2 - ab + b^2$ gives a Euclidean algorithm on $\mathbb{Z}[\omega]$ where $\omega = \frac{-1+\sqrt{3}i}{2}$.

Q 7. Let S be any ring with unity. In the ring $M_2(S)$, give examples of left ideals which are not of the form $M_2(I)$ for any left ideal I of S.

Q 8. Find all maximal ideals of $\mathbb{Z}[X]$ containing 4.

Q 9. Let $\theta : \mathbf{C}[X, Y] \to \mathbf{C}[T]$ be the ring homomorphism given by $X \mapsto T^2, Y \mapsto T^3$. Prove that Ker $\theta = (X^3 - Y^2)$.

Q 10. Let M be a left R-module over a commutative ring with unity, and let N be a submodule. If N and M/N are finitely generated, prove that M is finitely generated. Further, give an example of a free module over $\mathbb{Z}$ which has two minimal spanning sets of different cardinalities.

Q 11. Define the Jacobson radical $Jac(R)$ of a commutative ring R with unity and prove that $x \in Jac(R)$ if and only if $1 + xy$ is a unit for all $y \in R$.

Q 12. Define the companion matrix $C(f)$ of a monic polynomial $f \in K[X]$ for a field K. Prove that its characteristic polynomial is f. Further, if deg $f = 2$, prove by direct calculation that $C(f)$ is conjugate to its transpose.

Q 13. Determine all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[X]/(X^3 + X^2 + cX + 1)$ is a field.

Q 14. Let A be a commutative ring with unity. If I is an ideal which is maximal with respect to the property of not being principal, prove that I is a prime ideal. Further, if I is such an ideal, prove that A/I is a principal ideal ring.

Q 15. Let A be a commutative ring with unity, I be an ideal and $P_1, \cdots, P_m$ be prime ideals such that $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_m$. Then show that $I \subseteq P_i$ for some i.

Q 16. Let A be a commutative ring with unity. If I is an ideal which is maximal with respect to the property of not being principal, prove that I is a prime ideal. Further, if I is such an ideal, prove that A/I is a principal ideal ring.
Q 17. Let A be a commutative ring with unity, I be an ideal and $P_1, \cdots, P_m$ be prime ideals such that $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_m$. Then show that $I \subseteq P_i$ for some i.
Q 18. Show that the ring $C[0,1]$ is not Noetherian.
Q 19. Show that if $f = \sum_{i=0}^{m} a_i X^i, g = \sum_{j=0}^{n} b_j X^j \in (\mathbb{Z}/2^{100}\mathbb{Z})[X]$ are such that $fg = 0$, then $a_i b_j = 0$ for all i, j.
Q 20. Show that there is no commutative ring A with unity such that $A[X]$ is isomorphic to the ring of integers.
Q 21. If A is an integral domain, and I, J are ideals such that IJ is a principal ideal, prove that I, J are finitely generated.
Q 22. Let A be a local ring with the maximal ideal $\mathbf{m}$. Let M be a finitely generated A-module and $x_1, \cdots, x_n \in M$ be elements such that $M/\mathbf{m}M$ is generated as an $A/\mathbf{m}$-module by the images of the x_i's. Then prove that M is generated by the x_i's.
Hint: You may use the NAK lemma.
Q 23. Let A be a commutative ring with unity. and M be a finitely generated A-module. If $\theta : M \to M$ is an onto A-module homomorphism, prove that θ is $1-1$ as well.
Q 24. Let A be a commutative ring with unity.
(i) If I, J are ideals such that there exists an onto A-module homomorphism from A/I to A/J, prove that $I \subseteq J$.
(ii) If an ideal P is free as an A-module, prove that P must be principal.
Q 25. Prove that $\mathbf{Z} + X\mathbf{Q}[X]$ is not a UFD.
Q 26. Let $R = End(V)$, where V is a countably infinite-dimensional vector space. Elements of R can be regarded as column-finite matrices (that is, each column has only finitely many non-zero entries). The matrices x, y, u, v in R are defined as follows:
$1 = x_{2n-1,n}\ \forall\ n$, and $x_{ij} = 0$ for other i, j's;
$1 = y_{2n,n}\ \forall\ n$, and $y_j = 0$ for other i, j;
$u = x^t, v = y^t$.
Demonstrate how these matrices provide an isomorphism between left R-modules R and R^2.
Q 27. Let M be the $\mathbb{Z}[i]$-module given as the quotient of the free module $\mathbb{Z}[i]^3$ modulo the relations $f_1 = (1,3,6), f_2 = (2+3i, -3i, 12-18i), f_3 = 2-3i, 6+9i, -18i)$. Find the cardinality of M.
Q 28. Prove that two matrices $A, B \in M_n(\mathbb{C})$ are conjugate if and only

if the matrices $(zI_n - A)^k$ and $(zI_n - B)^k$ are conjugate for all $z \in \mathbb{C}$ and positive integer k.

Q 29. Are the matrices $B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ similar?

Hint: You may use results on rational canonical forms.

Q 30. Let R be a ring with unity and let I be a left ideal. Prove that $S = \{r \in R : Ir \subseteq I\}$ is a subring containing I as a 2-sided ideal. Further, show that the ring $End_R(R/I)$ of left R-module endomorphisms of R/I is isomorphic to the opposite of the ring S/I.