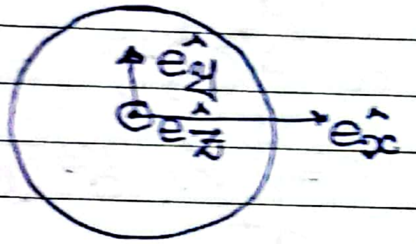
P1 (AZ, 2.5) (7 points)

The mechanical angular momentum $\vec{L}$ follows

$$\frac{d\vec{L}}{dt} = \vec{T}_{ext} \quad (\vec{T} \text{ is the torque})$$
$$= \vec{r} \times \vec{F}_{ext}$$

$$\vec{B}(x,y,t) = \Phi(t)\, \delta(x)\, \delta(y)\, \hat{e}_z$$

Consider a circle of radius r ~~at~~ parallel to $x-y$ plane ~~(in fact,~~



$$\oint_{C} \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

Which direction does $\vec{E}$ point?

In fact $\vec{E} = |\vec{E}|\, \hat{e}_\varphi$

$$\Rightarrow \quad 2\pi r E_\varphi = -\frac{d\Phi}{dt}$$

i.e., $$\vec{E} = \frac{-1}{2\pi r} \frac{d\Phi}{dt} \hat{e}_\varphi$$

$$\vec{F}_{ext} = q\vec{E} = \frac{-q}{2\pi r} \frac{d\Phi}{dt} \hat{e}_\varphi$$

$$\vec{T}_{ext} = \vec{r} \times \vec{F}_{ext}$$
$$= r\,\hat{e}_r \times \left(\frac{-q}{2\pi r}\right) \frac{d\Phi}{dt} \hat{e}_\varphi$$
$$= -\frac{q}{2\pi} \frac{d\Phi}{dt} \hat{e}_z$$
$$= \frac{d\vec{L}}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\vec{L} + \frac{q\phi}{2\pi}\hat{e}_z\right) = 0$$

i.e., $\vec{L} + \frac{q\phi}{2\pi}\hat{e}_z$ is constant of motion.

P2 (AZ 2.6, 4+6+3 = 13 points)

Given : $\rho(\vec{r},t) = \sum_{k=1}^{N} q_k(t)\, \delta^3(\vec{r}-\vec{r}_k)$

$$\frac{\partial\rho(\vec{r},t)}{dt} = \sum_{k=1}^{N} \frac{dq_k(t)}{dt}\, \delta^3(\vec{r}-\vec{r}_k)$$

We must find a $\vec{j}(\vec{r},t)$ s.t.

$$\vec{\nabla}\cdot\vec{j} = -\sum_{k=1}^{N} \frac{dq_k(t)}{dt}\, \delta^3(\vec{r}-\vec{r}_k)$$

But, $\delta^3(\vec{r}-\vec{r}_k) = \frac{+1}{4\pi}\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}\right)$

$$\therefore \vec{\nabla}\cdot\vec{j} = -\sum_{k=1}^{N} \frac{dq_k(t)}{dt} \times \left[\frac{1}{4\pi}\vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}\right)\right]$$

$\Rightarrow \vec{j}(\vec{r},t)$

$$= \frac{-1}{4\pi}\sum_{k=1}^{N}\left(\frac{dq_k(t)}{dt}\right)\left(\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}\right)$$

(b) Given, $\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N} q_k(t)\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}$

$\because \vec{\nabla}\cdot\left(\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}\right) = 4\pi\,\delta^3(\vec{r}-\vec{r}_k),$

$$\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \frac{1}{\epsilon_0}\sum_{k=1}^{N} q_k(t)\,\delta^3(\vec{r}-\vec{r}_k)$$

$$= \frac{\rho(\vec{r},t)}{\epsilon_0}$$

($\vec{E}(\vec{r},t)$ satisfies Gauss' Law).

$\because \vec{\nabla}\times\left(\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}\right) = 0$

$\therefore$ $$\vec{\nabla}\times\vec{E}(\vec{r},t) = 0$$

$$\Rightarrow -\frac{\partial\vec{B}(\vec{r},t)}{\partial t} = 0 \quad \forall\vec{r},$$

$$\Rightarrow \vec{B}(\vec{r},t) \equiv \vec{B}(\vec{r})$$

$\vec{B}(\vec{r})$ will remain the same over time at all $\vec{r}$.

i.e., $\vec{B}(\vec{r},t) = \vec{B}(\vec{r},t=0)\ \forall\vec{r}$

$\therefore \vec{B}(\vec{r},t) \equiv 0$;

. This trivially satisfies $\vec{\nabla}\cdot\vec{B} = 0$

$$\Rightarrow \vec{\nabla}\times\vec{B} = 0$$

$$= \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \frac{\mu_0}{4\pi}\sum_{k=1}^{N}\left(\frac{dq_k(t)}{dt}\right)\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}$$

$$\left[\because c^2 = \frac{1}{\mu_0 \epsilon_0}\right]$$

$$= -\mu_0 \vec{j}$$

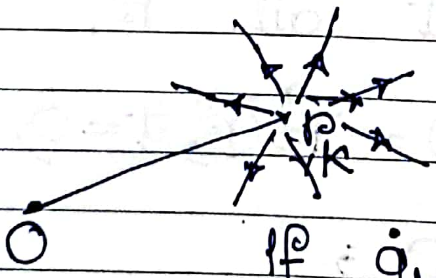
$\therefore$ The Ampere – Maxwell Law is also satisfied by $\vec{B}(\vec{r},t) \equiv 0$.

(c)

$$\vec{j} = \frac{-1}{4\pi}\sum_{k=1}^{N}\left(\frac{dq_k}{dt}\right)\frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}$$

$$= \sum_{k=1}^{N}\vec{j}_k(\vec{r},t)$$

$$\vec{j}_k(\vec{r},t) = \frac{-1}{4\pi}\,\dot{q}_k(t)\,\frac{\hat{e}_{\vec{r}-\vec{r}_k}}{|\vec{r}-\vec{r}_k|^2}$$

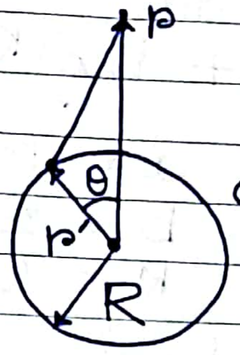


radial flow of charges from each $\vec{r}_k$.

if $\dot{q}_k(t) > 0$, flow is inward to $\vec{r}_k$;

if $\dot{q}_k(t) < 0$, flow outward from $\vec{r}_k$;

P3. AZ 2.14 (10 points)



$$\sigma = \frac{Q}{4\pi R^2}$$

- There is full azimuthal symmetry of σ, so all we need to do is find $\varphi(\vec{r})$ on z-axis and then replace z by r.

$$d\varphi(\vec{r}) = \frac{dq(\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|^{(1+\eta)}}$$

$$= \frac{\sigma R^2\, d\Omega}{4\pi\epsilon_0 |r^2 + r'^2 - 2rr'\cos\theta|^{\kappa}}$$

where $\kappa = (1+\eta)/2$

$$d\Omega = \sin\theta\, d\theta\, d\varphi; \quad r' = R.$$

$$\therefore \quad \varphi(\vec{r}) = \int d\varphi(\vec{r})$$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\theta) \times \left(\frac{1}{(r^2+R^2-2rR\cos\theta)^{\kappa}}\right)$$

The integral over θ is of the form $\int_{-1}^{+1} \frac{dx}{(a+bx)^k}$

$$a = r^2 + R^2$$
$$b = 2rR.$$
$$k = (1+\eta)/2$$

which ~~of~~ is best done by

$$u = a+bx$$
$$du = b\,dx$$

to write it in the form

$$\int_{u_{min}}^{u_{max}} \frac{du}{u^k} \times \left(\frac{1}{b}\right)$$

$$\Rightarrow \varphi(r) = \frac{Q}{4\pi\epsilon_0} \frac{1}{1-\eta} \cdot \frac{1}{2Rr} \times \left\{ |r+R|^{1-\eta} - |r-R|^{1-\eta} \right\}$$

· If $\eta \to 0$

(i) $r > R \Rightarrow |r+R| = r+R$
$$|r-R| = r-R$$
$$\Rightarrow |r+R| - |r-R| = 2R$$
$$\Rightarrow \varphi(r) = \frac{Q}{4\pi\epsilon_0 r}$$

(ii) $r < R \Rightarrow |r-R| = R-r$

$$\Rightarrow \varphi(r) = \frac{Q}{4\pi\epsilon_0 R}$$

as expected in the $\eta \to 0$ Coulomb case.

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P4 (AZ 3.2, 5+5=10 points)

(a) $$E_k = c_k + D_{jk} r_j$$

$$\partial_k E_k = 0 \quad [\because \rho = 0]$$

$$\Rightarrow D_{jk} \partial_k r_j = 0$$

$$\Rightarrow D_{jk} \delta_{kj} = 0 \Rightarrow D_{kk} = 0$$

$\Rightarrow D$ is traceless.

Also, $$\epsilon_{pqk} \partial_q E_k = [\vec{\nabla} \times \vec{E}]_p = 0$$

$$\Rightarrow \epsilon_{pqk} D_{jk} \underbrace{\partial_q r_j}_{\delta_{qj}} = 0$$

$$\Rightarrow \epsilon_{pjk} D_{jk} = 0$$

Now j & k are 'dummy' indices

$$\therefore \epsilon_{pjk} D_{jk} + \epsilon_{pkj} D_{kj} = 0 + 0 = 0$$

But $\epsilon_{pjk} = -\epsilon_{pkj} \; \forall \{j,k\}$

$$\Rightarrow (D_{jk} - D_{kj}) \epsilon_{pjk} = 0$$

which is possible iff $D_{jk} = D_{kj} \; \forall \{j,k\}$.

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(b) $\because E_k = -\partial_k \varphi$

$$\Rightarrow \varphi = A - C_j r_j - \frac{1}{2} D_{ab} r_a r_b$$

(the constants (or choice of $\vec{r}$ @ $\varphi(\vec{r}) = 0$) are lumped into A.

P5 (AZ 3·6) 10 points

Let $\rho'(\vec{r})$ cause $\vec{E}'(\vec{r})$.

At $\vec{r}$, $d\vec{F}$ (on $\rho(\vec{r})$) $= \rho(\vec{r}) \vec{E}'(\vec{r})$

$\therefore \vec{T}$ (on ρ due to $\vec{E}'$ {caused by ρ'})

$$= + \int_V dV \left[\vec{r} \times \rho(\vec{r}) \vec{E}'(\vec{r}) \right]$$

$$= - \int_V dV \left[\vec{r} \times \rho(\vec{r}) \{ +\vec{\nabla} \varphi'(\vec{r}) \} \right]$$

$$\therefore T_i = -\epsilon_{ijk} \int_V dV \, r_j \, \rho(\vec{r}) \, \partial_k \varphi'(\vec{r})$$

Shift the derivative to $r_j \rho(\vec{r})$, integrate by parts, and ignore the surface term since $\varphi'(\vec{r}) \to 0$ ~~and~~ as $S (\equiv \partial V) \to \infty$.

$$\therefore T_i = \epsilon_{ijk}\int_V dV\, \phi'(\vec{r})\, \partial_k\left(r_j \rho(\vec{r})\right)$$

$$\epsilon_{ijk}\,\partial_k(r_j\rho)$$
$$= \epsilon_{ijk}\,\delta_{jk}\,\rho + \epsilon_{ijk}\, r_j\, \partial_k \rho$$

$$\epsilon_{ijk}\,\delta_{jk} = 0 \quad \forall j$$

$$\Rightarrow T_i = \epsilon_{ijk}\int_V dV\, \phi'(\vec{r})\, r_j\, \partial_k \rho(\vec{r})$$

Now, $\phi'(\vec{r}) = \int_{V'} dV' \dfrac{\rho'(\vec{r}')}{|\vec{r} - \vec{r}'|} \cdot (4\pi\epsilon_0)^{-1}$

$$\therefore T_i = \epsilon_{ijk}\int dV \int dV'\ \frac{1}{4\pi\epsilon_0}\ \frac{\rho'(\vec{r}')\, r_j\, \partial_k \rho(\vec{r})}{|\vec{r} - \vec{r}'|}$$

Yet again, shift the derivative to $v_j(\vec{r}) = \dfrac{r_j}{|\vec{r} - \vec{r}'|}$, extend both integrals to all space, integrate by parts and ignore the surface term (it vanishes)

$$\therefore T_{i} = \frac{-1}{4\pi\epsilon_0}\int d^3r \int d^3r' \, \rho(\vec{r})\,\rho(\vec{r}') \times \epsilon_{ijk}\,\partial_k \left[\frac{r_j}{|\vec{r}-\vec{r}'|}\right]$$

Now, $\partial_k \left[\frac{r_j}{|\vec{r}-\vec{r}'|}\right]$

$$= \frac{\delta_{jk}}{|\vec{r}-\vec{r}'|} - \frac{r_j(r_k - r_k')}{|\vec{r}-\vec{r}'|^3}$$

$\epsilon_{ijk}\,\delta_{jk} = 0$, so term 1 vanishes

$$\therefore T_i = \frac{-1}{4\pi\epsilon_0}\int d^3r \int d^3r' \, \rho(\vec{r})\rho(\vec{r}')$$

$$\times \; \epsilon_{ijk}\left[\frac{r_j r_k}{|\vec{r}-\vec{r}'|^3} - \frac{r_j r_k'}{|\vec{r}-\vec{r}'|^3}\right]$$

$$\epsilon_{ijk} r_j r_k = (\vec{r}\times\vec{r})_i$$

$$= 0$$

$$\therefore T_i = \frac{1}{4\pi\epsilon_0}\int d^3r \int d^3r' \, \rho(\vec{r})\rho(\vec{r}') \frac{(\vec{r}\times\vec{r}')_i}{|\vec{r}-\vec{r}'|^3},$$

(Q.E.D.)

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P6 (10 points)

This is identical to P3 for $\eta \to 0$. $\therefore$ The method and the solution should be identical to its $\eta \to 0$ limit.

$$\phi(r>R) = \frac{4\pi R^2 \sigma}{4\pi \epsilon_0 r}$$

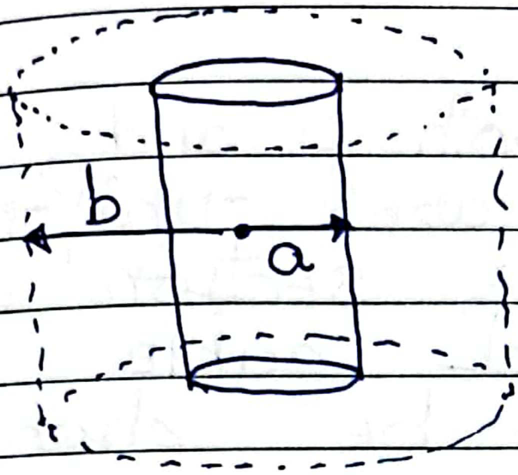
$$= \frac{\sigma R^2}{\epsilon_0 r}$$

$$\phi(r<R) = \frac{4\pi R^2 \sigma}{4\pi \epsilon_0 R}$$

$$= \frac{\sigma R}{\epsilon_0}$$

- Note $\phi(r<R)$ is a const., as it should be $\because \vec{E} = 0$ for $r<R$.

P7 (7+5+3 = 20 points)



- We work in cylindrical ~~solutio~~ co-ordinates (s, ϕ, z)

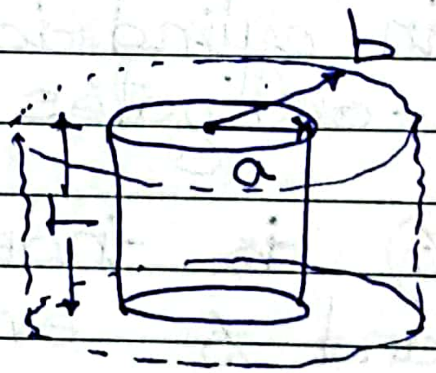
$\because$ $\rho(s, \phi, z)$ is independent of ϕ and z everywhere, so is $\vec{E}$

$$\therefore \vec{E}(s) = E_s(s)\,\hat{e}_s + E_\phi(s)\,\hat{e}_\phi + E_z(s)\,\hat{e}_z$$

- We've discussed this situation is class, and, we know that $\forall s$, $\vec{E}(s) = E_s(s)\,\hat{e}_s$.

Also, note that ρ and σ are not independent.

$\because$ Since both ρ and $\sigma = -|\sigma|$ are uniform, any finite section of the coaxial cable must also be charge neutral.



$$Q_{volume} = \rho \times \text{Volume of the 'a'-cylinder}$$
$$= \rho \times \pi a^2 L$$

$$Q_{surface} = -|\sigma| \times \text{Surface area of the b-cylinder}$$
$$= -|\sigma| \times 2\pi b L$$

$\therefore \rho \times \cancel{\pi} a^2 \cancel{L} + (-|\sigma|) \times 2\cancel{\pi} b \cancel{L} = 0$

$\Rightarrow \rho a^2 = |\sigma| \times 2b$

$\Rightarrow |\sigma| = \dfrac{\rho a^2}{2b}$

(i) We simply insert Gaussian surfaces at various region.

Region I : (s<a)

$$\int_{z=-L/2}^{L/2} \int_{\phi=0}^{2\pi} E_s(s)\, \hat{e}_s \cdot s\, d\phi\, dz\, \hat{e}_s = \int_{S} \vec{E} \cdot \vec{dS} = Q_{enc}/\epsilon_0$$

RHS : $\dfrac{Q_{enc}}{\epsilon_0} = \dfrac{\rho \pi s^2 L}{\epsilon_0}$

$\therefore E_s(s) \times \cancel{s} \times 2\cancel{\pi} \times \cancel{L} = \dfrac{\rho \cancel{\pi} s^{\cancel{2}} \cancel{L}}{\epsilon_0}$

$\Rightarrow E_s(s; s<a) = \dfrac{\rho s}{2\epsilon_0}$

$$\therefore \vec{E}_s(s; s<a) = \frac{\rho s}{2\epsilon_0}\hat{e}_s \quad \ldots (1)$$

For $a < s < b$,

LHS

$$= \oint E_s(s) \times 2\pi s L$$

(as always)

$$\text{RHS} = \frac{\rho \pi a^2 L}{\epsilon_0}$$

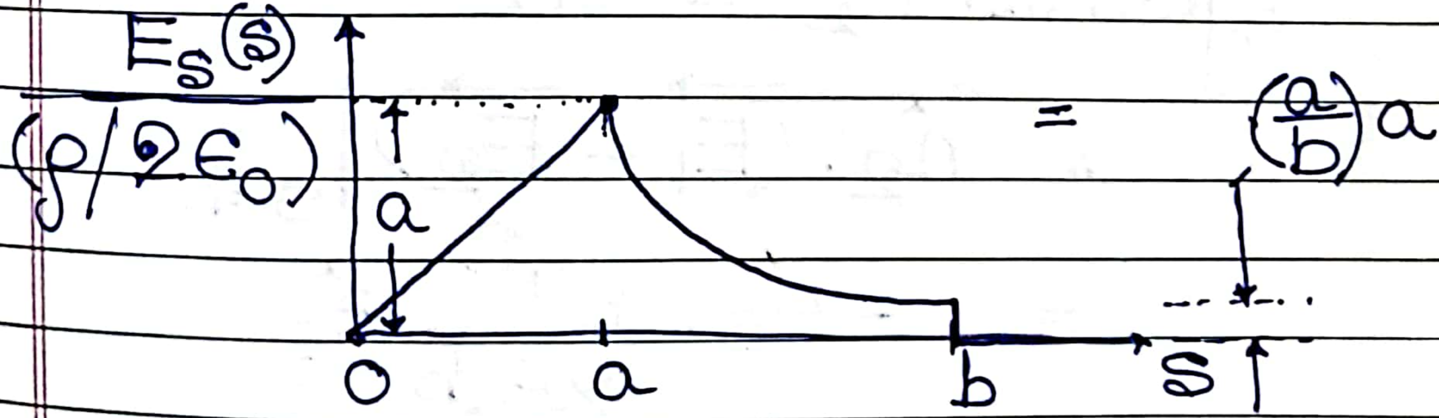
$$\Rightarrow E_s(s)(a<s<b) = \frac{\rho a^2}{2\epsilon_0 s}$$

$$\Rightarrow \vec{E}_s(a<s<b) = \frac{\rho a^2}{2\epsilon_0 s}\hat{e}_s \quad \ldots (2)$$

part (iii) of Sol: Q. i

Note (1) & (2) match at $s=a$, as it should because no surface charge at $s=a$.

For $s>b$, there is $\vec{E}=0$, since the co-axial cable has $Q_{enc}=0$ for any finite cylindrical G.S.



For the second interface, let $\hat{e}_s$ be the outward normal $\hat{n}_{in\to out} \equiv \hat{n}_1$

B.C: $\vec{E}_{in}(s=b)\cdot\hat{n}_{in\to out}$

$\therefore$ Region I $\equiv s<b$.

$$\vec{E}_1 = \frac{\rho a^2}{2\epsilon_0 s}\hat{e}_s$$

Region II: $\vec{E}_2 = 0$

$$\therefore \hat{n}_2 \cdot (\vec{E}_1 - \vec{E}_2)\Big|_{s=b}$$

$$\equiv -\hat{e}_s \cdot \frac{\rho a^2}{2\epsilon_0 b}\hat{e}_s$$

$$= \frac{-|\sigma|}{\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

$$\left(\because \frac{\rho a^2}{2b} = |\sigma|\right)$$

satisfying M.C. at $s=b$

(Part (iii), second part)

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(ii) The potential is trivially obtained by integrating. Take a path from $s'=0$ to $s'=s$ $(s<a)$ along any ϕ (e.g. $\phi=0$) and on the plane $z=0$.

$$-\int_{s'=0}^{s'=s} \vec{E}\cdot d\vec{l} = \phi_f - \phi_{initial}$$

$$= -\int_{s'=0}^{s'=s} \frac{\rho s'}{2\epsilon_0}\hat{e}_s \cdot ds'\hat{e}_s$$

$$= \frac{-\rho s'^2}{2\epsilon_0}\Big|_0^s = \frac{-\rho s^2}{2\epsilon_0} - \phi(s=0)$$

- Putting $\phi(s=0)=0$

$$\phi(s) = \frac{-\rho s^2}{2\epsilon_0} \quad (s<a)$$

[The -ve potential is an artifact of our choosing $\phi(0)=0$. Physically, a +ve charge in $s<a$ will

accelerate outward i.e., increasing s, i.e., toward lower potential.]

For $a<s<b$

$$-\int_{s'=a}^{s'=s} \vec{E}\cdot d\vec{l}' = \varphi(s) - \varphi(a)$$

$$\Rightarrow \frac{\rho a^2}{2\epsilon_0}\int_{s'=a}^{s'=s} \frac{\hat{e}_s}{s'}\cdot ds'\,\hat{e}_s$$

$$= \frac{\rho a^2}{2\epsilon_0} \ln s' \Big|_a^s$$

$$= \frac{\rho a^2}{2\epsilon_0} \ln\left(\frac{s}{a}\right)$$

$$= \varphi(s) + \frac{\rho a^2}{2\epsilon_0}$$

~~For s = a,~~

$$\therefore \varphi(s) = \frac{\rho a^2}{2\epsilon_0}\left[\ln\left(\frac{s}{a}\right) - 1\right]$$

$\varphi(a) = -\frac{\rho a^2}{2\epsilon_0}$ from both sides, satisfying continuity.

For $s > b$, $\varphi(s) = \frac{\rho a^2}{2\epsilon_0}\left[\ln\left(\frac{b}{a}\right) - 1\right]$

i.e., const., $\because \vec{E} = 0$.

—x—