Statistics-III – Assignment 2

1. Let $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_{p+p}\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma'_{XY} & \Sigma_Y \end{pmatrix}\right)$ and define: $U = X + Y$, $V = X - Y$. When is U independent of V?

2. Let Z_1, Z_2, Z_3 be i.i.d. $N(0,1)$ and $0 < \rho < 1$. Define $X_1 = Z_1$, $X_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$ and $X_3 = \rho X_2 + \sqrt{1-\rho^2} Z_3$. What is the joint distribution of $(X_1, X_2, X_3)'$?

3. Let Z_1, Z_2 be i.i.d. $N(0,1)$ and $0 < \rho < 1$. Define $X_1 = Z_1$, $X_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$. Find X_3 such that

$$\text{Cov}\,(X_1, X_2, X_3) = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}.$$

4. Let $\mathbf{Y} \sim N_n(\theta, \sigma^2 I_n)$, and let $\mathbf{X} = A\mathbf{Y}$, $\mathbf{U} = B\mathbf{Y}$ and $\mathbf{V} = C\mathbf{Y}$, where A, B and C are all $r \times n$ matrices of rank $r < n$. If $Cov(\mathbf{X}, \mathbf{U}) = 0$ and $Cov(\mathbf{X}, \mathbf{V}) = 0$, show that $\mathbf{X}$ is independent of $\mathbf{U} + \mathbf{V}$.

5. Let $Z \sim N(0,1)$. Define

$$Y = \begin{cases} Z & \text{if } |Z| > c; \\ -Z & \text{if } |Z| \le c. \end{cases}$$

Show that (Z, Y) has a joint distribution under which the marginal distributions are normal, but the joint distribution is not bivariate normal.