Homework & Assignment-2

1. Let X be a metric space, $x \in X$, $r > 0$. Is it true that $\overline{B(x,r)} = \{y \in X \mid d(x,y) \leq r\}$?

2. X be a metric space, $U \subseteq X$ open, $A \subseteq X$. Prove that $U \cap A = \phi \iff U \cap \bar{A} = \phi$.

3. X a metric space. Prove that $A \subseteq X$ is dense $\iff$ the only closed set ~~properly~~ $\supseteq A$ is X $\iff$ the only open set disjoint from A is ϕ $\iff$ A intersects every nonempty open set $\iff$ A intersects every open ball.

$\boxed{4}$. Consider $X = \mathcal{C}([0,1], \mathbb{R})$ with the norm $\|f\| = \int_0^1 |f(x)|dx$ giving the metric $d(f,g) = \|f-g\|$. Define $f_n \in X$ by $f_n(x) = 1$ for $x \in [0, 1/2]$, $f_n(x) = -2^n(x - 1/2)$ for $x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}]$ and $f_n(x) = 0$ for $x \in [1/2 + 1/2^n, 1]$. Prove that (f_n) is a Cauchy sequence in X which is not convergent.

5. Give an example to show that the diameter sequence converging to zero in Cantor's intersection theorem is essential, i.e. cannot be dropped from the hypothesis.

6. A closed subset of X is nowhere dense $\iff$ its complement is dense.

7. Prove that the Cantor set is nowhere dense.

8. Let (X,d) be a metric space, $x_0 \in X$. Show that $f_{x_0}: X \to \mathbb{R}$, $f_{x_0}(x) = d(x_0, x)$ is continuous. Is f_{x_0} uniformly continuous?

9. Let X, Y be metric spaces, $A \subseteq X$ nonempty. Show that for $f, g: X \to Y$ continuous, $f(x) = g(x)\ \forall x \in A$ $\Rightarrow f(x) = g(x)\ \forall x \in \bar{A}$.

10. Let X be any nonempty set and $\mathcal{B}(X)$ be the $\mathbb{R}$-vector space of all bounded functions on X. Let $f \in \mathcal{B}$, $\|f\| := \operatorname{Sup}\{|f(x)|, x \in X\}$. Prove that $(\mathcal{B}(X), d)$, for $d(f,g) := \|f-g\|$, is a complete metric space.

11. What the completion of (X, d) in Problem 4?

12. Let X = the set of positive integers with discrete metric. Let $\mathcal{C}(X, \mathbb{R})$ be the space of bounded (continuous) functions on X. Show that $\mathcal{C}(X,\mathbb{R})$ is not separable.

$\boxed{13}$. Replace X in (12) by an arbitrary discrete metric space. Prove that $\mathcal{C}(X,\mathbb{R})$ is separable $\Longleftrightarrow$ X is finite.

14. A function $f : X \times Y \to Z$ of topological spaces is jointly continuous in x & y if f is continuous; we say f is continuous in x, if for any $y \in Y$, the map $: X \to Z$ given by $x \mapsto f(x,y)$ is continuous. Similarly we define continuity of f in y.

- Assume all X, Y, Z are metric spaces. Show that f is jointly continuous $\Longleftrightarrow$ $x_n \to x$, $y_n \to y$ implies $f(x_n, y_n) \to f(x,y)$.
- f is jointly continuous $\Rightarrow$ f is continuous in each variable separately. Show that converse of this is false.

Please submit solutions of Problems 4 & 13 as Assignment-2, both carry 2 points.