Homework-3

1. For a metric space (X,d), the sphere with centre p & radius r is given by $S(p,r)=\{x\in X \mid d(p,x)=r\}$. When $X=\mathcal{C}([0,1],\mathbb{R})$, prove that for $f\in X$,
$$\partial B(f,r) = S(f,r).$$

2. i) Let X be a topological space, A, B subsets of X. Call A, B as <u>separated</u> if $\bar{A}\cap B = A\cap \bar{B} = \phi$. Prove that X is disconnected $\Leftrightarrow \exists\, A, B \subseteq X$, both nonempty & separated, $X = A\cup B$.

   ii) Let (X,d) be a metric space, $p\in X$, $\delta>0$. Let $A = B(p,\delta)$ and $B=\{x\in X \mid d(p,x)>\delta\}$. Prove that A and B are separated.

   iii) Prove that a connected metric space with at least two points is necessarily uncountable.

3. If $A\subseteq X$ is connected then explore if A° is connected.

4. i) Call a subset E of a metric/topological space X as perfect if E is closed & every point of E is a limit point of E.
Let X be a separable metric space, $C\subseteq X$ closed. Prove that C is the union of a (possibly ϕ) perfect set and an atmost countable set.

ii) Every countable closed subset of $\mathbb{R}^n$ has isolated points.

iii) A point p of a metric space X is a condensation point of a set $E \subseteq X$ if every neighbourhood of p contains uncountably many points of E.

→ Let $E \subseteq \mathbb{R}^n$ be uncountable & P be the set of all condensation points of E. Prove that P is perfect & $P^c \cap E$ is at most countable.

[Hint: Let $\{V_i\}$ be a countable basis of $\mathbb{R}^n$ & W be the union of the V_i that have $V_i \cap E$ at most countable. Show that $P = W^c$.]

iv) Is it true that $D \subseteq \mathbb{R}^n$ is discrete then D is countable?

5. Prove the cardinality of the set of all connected components of a top. space, is an invariant of that space.

$\boxed{6}$. Consider $X = \mathcal{C}(S^n, \mathbb{R}^m)$ with the 'sup' metric. Prove that X is path connected.

$\boxed{7}$. i) Prove that: the Cantor set $\mathcal{C}_3$ is non empty!

ii) for $m, k \geq 1$, the intervals $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ are disjoint from $\mathcal{C}$.

iii) Any interval (a, b) contains a sub-interval of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ for some $k, m \geq 1$.

iv) $\mathcal{C}$ contains no positive length inteval.

v) $\mathcal{C}$ is uncountable.

vi) Prove $\mathcal{C}$ is perfect following the steps:
let $x \in \mathcal{C}$ & J an inteval (open), $x \in J$ & I_n be that (closed) interval of F_n that contains x.
→ For large n, $I_n \subset J$. Let x_n be an end point of I_n such that $x_n \neq x$.
→ $x_n \in \mathcal{C}$
→ x is a limit point of $\mathcal{C}$.

Here $F_0 = [0, 1]$, $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$,
$F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
⋮
F_{i+1} is obtained from F_i by removing middle third open interval of each interval present in F_i; so $F_0 \supset F_1 \supset \cdots$ &
$\mathcal{C} = \bigcap_i F_i$

Please submit Solutions of Problem 6 & Problem 7 as Assignment-3.