Homework

1. Cone vs Geometric cone: Consider $X = \{(n,0) \mid n \in \mathbb{Z}^{>0}\} \subseteq \mathbb{R}^2$ and C be the subspace of $\mathbb{R}^2$ obtained by joining every $x \in X$ to $p = (0,1)$ by a line segment. Let $C(X) = (X \times [0,1]) / (X \times \{1\})$ be the cone over X.

→ Prove that there is a continuous bijection : $C(X) \to C$, but $C(X)$ is not homeomorphic to C.

In the above, X is not compact.

→ Let $X \subseteq \mathbb{R}^n$ be compact, $C(X)$ the topological cone over X and C the geometric cone : $\{(1-t)x + tp \mid x \in X, p \notin X \text{ fixed } \& \ t \in [0,1]\}$.

Prove then $C(X)$ is homeomorphic to C.

2. Let X, Y be topological spaces, $X \xrightarrow{f} Y$ a continuous map, the $C(f) : C(X) \to C(Y)$, $[(x,t)] \longmapsto [(f(x),t)]$ is continuous.

Group actions: We say a group G acts on a topological space Y evenly if any $y \in Y$ has an open neighbourhood U such that $\forall$ $g \neq h \in G$, $g.U \cap h.U = \emptyset$.

(i) • The group μ_n of all n^{th} roots of unity in $\mathbb{C}$ acts on $\mathbb{C}$ by left multiplication.

→ Show that this action is not even.

→ The same action of μ_n on $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ is even.

(ii) • Let G act evenly on Y. Consider the orbits of points in Y under this action. Prove that they are all discrete.

(iii) • Let G be the subgroup of the group of all self homeomorphisms of $\mathbb{R}^2$ generated by the translation $(x, y) \mapsto (x+1, y)$ and the map $(x, y) \mapsto (-x, y+1)$.

Prove that this is an even action of G on $\mathbb{R}^2$. Also show that $\mathbb{R}^2/G$ is the Klein bottle.

(iv) • Let G be a finite group action fixed point freely on a Hausdorff Space Y, i.e. $g.y = y$ for some $g \in G$ & $y \in Y \Rightarrow g = e$.

Prove that such an action is even.

(v) • We discussed an action of μ_n on the Complex sphere $S^{m-1}_{\mathbb{C}} := \{(z_1, \cdots, z_m) \mid |z_1|^2 + \cdots + |z_m|^2 = 1\}$, $\zeta \cdot (z_1, \cdots, z_m) := (\zeta z_1, \cdots, \zeta z_m)$. Prove this is an even action.

* Probems (i), (iii), (iv), (v) constitute Assignment-5: Submit by 15^{th} April.