Homework practice problems

1. Let p be a prime and n be a nonzero integer. Write $n = p^{\nu_p(n)} m$ where m is coprime with p. Extend the function ν_p to the set of nonzero rationals by the formula $\nu_p(\pm\frac{r}{s}) = \nu_p(r) - \nu_p(s)$. Finally, define $\rho(x, y) = p^{-\nu_p(x-y)}$ when $x \neq y$ and $\rho(x, x) = 0$ for arbitrary $x, y \in \mathbb{Q}$. Prove that ρ is well defined and is a metric on $\mathbb{Q}$. This is the p-adic metric on $\mathbb{Q}$.

2. Let $S^n = \{(a_0, \cdots, a_n) \in \mathbb{R}^{n+1} | a_0^2 + \cdots a_n^2 = 1\}$ be the n-sphere in $\mathbb{R}^{n+1}$. Let $S_+^{n+1} := \{(a_0, \cdots, a_n) \in S^n,\ a_n > 0\}$ Prove that S_+^n is homeomorphic to the open unit ball in $\mathbb{R}^n$, where we identify $\mathbb{R}^n$ with the subspace $\{(a_0, \cdots, a_{n-1}, 0),\ a_i \in \mathbb{R}\}$ of $\mathbb{R}^{n+1}$. ()

3. **(i)** Let X, Y be a topological spaces. Treat X as the indexing set for copies of Y, so $Y_x = Y$ for every $x \in X$. Consider $\Pi_{x\in X} Y_x$ with product topology. Let $C(X, Y)$ be the set of all continuous maps $X \to Y$. Identify $C(X, Y)$ with a subset of $\Pi_{x\in X} Y_x$. Prove that the subspace topology on $C(X, Y)$ from $\Pi_{x\in X} Y_x$ coincides with the topology of pointwise convergence, for which a subbasis is given by the sets of the form $\{x_i,\ U_i\}_{i=1}^k := \{f \in C(X, Y) | f(x_i) \in U_i, i = 1, \cdots, k\}$, where $x_1, \cdots, x_k \in X$ and $U_1, \cdots, U_k$ are open subsets of Y.
(ii) Consider a sequence (f_n) in $C(X, Y)$ that converges to f in the topology of pointwise convergence on $C(X, Y)$. Then the sequence $(f_n(x))$ converges to $f(x)$ for every x.

4. Let X and Y be topological spaces. Consider all possible sets of the form

$$[K, U] := \{f \in C(X, Y) :\ f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset Y$ is open. The topology generated by these sets as a subbasis of open sets, is called the compact-open topology on $C(X, Y)$. Prove that when X is compact and Y is a metric space, the compact-open topology coincides with the topology of uniform convergence on the metric space $(C(X, Y), d)$, $d(f, g) = sup_{x\in X} d(f(x), g(x))$.

5. Show that the set of all isolated points of a second countable topological space is empty or countable. Hence show that any uncountable subset A of such a space must have at least one point which is a limit point of A.