Homework

1. Give an example of a topological group G, $A, B \subseteq G$, both closed, but AB not closed.
   - Show that if A is closed & B a compact subset of a topological group G, then AB is closed.
2. Let $G = \mathbb{R}$, $\alpha \in \mathbb{R}$ irrational. Prove that, for $A = \mathbb{Z}$, $B = \alpha\mathbb{Z}$, $A + B$ is not closed.
3. G be a topological group and $N \trianglelefteq G$ be a normal subgroup. Prove that G/N is discrete if and only if N is open.
4. The topological groups $\mathbb{C}^*$ and $(\mathbb{R}^{>0}, \cdot) \times S^1$ are isomorphic as topological groups (i.e. via a group isom, which is a homeomorphism)
5. The topological group $S^1 \times S^1$ is a topological quotient group of the topological group $\mathbb{C}^*$.
6. Let $D \subset GL_n(\mathbb{R})$ be the subgroup of all scalar matrices. Prove that $GL_n(\mathbb{R})/D$ is locally compact and Hausdorff top. group.
7. $f : G_1 \to G_2$ be a surjective continuous homomorphism of topological groups. Assume G_1 is compact & G_2 is Hausdorff. Prove that f is an open map.
8. Give an example $f : K \to Y$ continuous, K compact top space, Y Hausdorff, f onto, but not open.

9. Let $K \leq G$ be a compact subgroup of a top. group G. Show that the quotient map $G \to G/K$ is closed.

- Is the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \equiv S^1$ a closed map?

10. Prove that $O(n)$ is homeomorphic to $SO(n) \times \mathbb{Z}/2$. Are these isomorphic as top. grps?

11. Let G be a top. Hausdorff grp, $A, B \subseteq G$ compact subsets. Prove that AB is compact.

12. Prove that every nontrivial discrete subgrp of $(\mathbb{R}, +)$ is cyclic.

13. Let $A, B \in O(2)$, $\det A = 1$, $\det B = -1$. Show that $B^2 = I$ & $BAB^{-1} = A^{-1}$. Deduce that every discrete subgroup of $O(2)$ is cyclic or dihedral.

14. Show that the group of topological group automorphisms of S^1 is $\simeq \mathbb{Z}/2$.

15. T be an autom. of the top grp $(\mathbb{R}, +)$. Prove that $T(x) = x\,T(1)$ $\forall x \in \mathbb{R}$. Deduce that $\mathrm{Aut}_{\text{TOP-Gr}}((\mathbb{R}, +)) \simeq \mathbb{R} \times \mathbb{Z}/2$.