Misc. Problems

1. Prove that S^n is path connected for $n \geq 1$.
2. Prove, by giving explicit maps, $D^n/_{S^{n-1}}$ is homeomorphic to S^n, where D^n = Closed unit ball in $\mathbb{R}^n$ & $S^{n-1} = \partial D^n$.
3. Recognize the space obtained as $\mu/_{S^1}$, where μ = Möbius band & S^1 is its boundary circle.
4. Let $f: X \to Y$ be a quotient map, $A \subseteq X$ a subspace & let $f(A)$ have the induced topology from Y. Is $f|_A : A \to f(A)$ a quotient map?
5. Let $X \subseteq \mathbb{R}^2$ be the subspace given by

$$X = \bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\},$$ the Hawaiian earring. Let $Y = X \cup X^{op}$, where

$$X^{op} \subseteq \mathbb{R}^2 \text{ is the subspace } \{(x,y) \mid (x + 1/n)^2 + y^2 = 1/n^2\}.$$

Are X and Y homeomorphic? Explain.
6. Let Y be a fixed top. space, Λ an indexing set with cardinality at least $\aleph_0$. Let $Y_\alpha = Y$ $\forall \alpha \in \Lambda$ & let $Z = \prod_\alpha Y_\alpha$. Then for any indexing set Γ with cardinality $\leq$ cardinality of Λ & $Z_\beta = Z$ $\forall \beta \in \Gamma$, $\prod_{\beta \in \Gamma} Z_\beta$ is homeomorphic to Z.
7. Prove that the cube $\overbrace{I \times \cdots \times I}^{n}$ is a quotient of

the Cantor set.

8. Let $X_1 \subset X_2 \subset \cdots$ be a sequence of top spaces, where X_i is a closed subspace of X_{i+1} $\forall i$. Let $X = \bigcup_i X_i$ and define $U \subseteq X$ to be open in X if $X \cap U_i$ is open in X_i $\forall i$.

- Check this defines a topology on X, called the topology on X coherent with the X_i's.
- Each X_i has the subspace top from X.
- If each X_i is normal, show that X is normal.

9. Let X be metrizable. Prove TFAE

(i) X is bounded with respect to any metric d that gives the topology on X.

(ii) Every continuous $\varphi : X \to \mathbb{R}$ is bounded.

(iii) X is compact.

10. Write $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid z_0\bar{z}_0 + z_1\bar{z}_1 = 1\}$; p, q be relatively prime & $\mathbb{Z}/p = \langle g \rangle$. Define the lens space $L(p,q) := S^3/_{\mathbb{Z}/p}$, for the action $g \cdot (z_0, z_1) = (e^{2\pi i/p} z_0, e^{2\pi q i/p} z_1)$. Prove $L(2,1) \simeq \mathbb{RP}^3$. If $q - q'$ ~~divides~~ is $\equiv 0 \bmod p$, prove $L(p,q) \simeq L(p,q')$.