Indian Statistical Institute, Bangalore
B. Math.
Third Year, First Semester
Analysis on Graphs

Home Assignment I — Due Date : September 8, 2023
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(1) Let A be an $n \times n$ complex matrix. Show that the sum of $k \times k$ principal minors of A is same as the sum of products of eigenvalues of A, taken k at a time.

(2) (Star-graph) Fix $n \geq 2$. Consider the graph $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $E = \{\{1, j\} : 2 \leq j \leq n\}$. Write down the spectral decompositions of the adjacency matrix and the Laplacian matrix of G.

(3) Let $K = K_{p,q}$ be the complete bi-partite graph with p, q vertices. (i) Write down the incidence matrix of K (using suitable orientation); (ii) Write down the spectral decomposition of the adjacency matrix of K (iii) Write down the Laplacian of K. (iv) Compute all eigenvalues of the Laplacian of K.

(4) Let G be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. For $i, j \in V$, write $i \sim j$ (i is adjacent to j) if $\{i, j\} \in E$. Let D be the degree matrix and let A be the adjacency matrix of G. Then the *'sign-less Laplacian'* is defined as :

$$L_0(G) = D + A$$

Let $\{v_1, \ldots, v_n\}$ be the standard basis of $\mathbb{C}^n$. Show that
(i) $L_0(G)$ is positive;
(ii) $L_0(G) = \sum_{i \sim j}(v_i + v_j)(v_i + v_j)^*$.
(iii) $L(G) = \sum_{i \sim j}(v_i - v_j)(v_i - v_j)^*$.
(iv) Show that $L_0(G)$ is singular iff G is bipartite.

(5) Let Q be an $n \times m$ matrix with $n \leq m$. Let p, q be the characteristic polynomials of QQ^* and Q^*Q respectively. Show that $q(x) = p(x)x^{m-n}$. In particular, the spectrum of Q^*Q is same as that of QQ^* except for some additional zeros. (Hint: You may use the polar decomposition for Q.)

(6) Let G be the graph with $V = \{1, 2, 3, 4, 5\}$ and $\{i, j\}$ is an edge if either i divides j or j divides i. Compute the number of spanning trees of this graph.

(7) Show that if G is a connected graph with n vertices and $n - 1$ edges then G is a tree.

(8) Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. Then the complement graph G^c is defined as $V(G^c) = \{1, 2, \ldots, n\}$ and the edge set $E(G^c) = \{\{i, j\} : i \neq j, \{i, j\} \notin E(G)\}$. Compute the spectrum of the Laplacian of G^c in terms of the spectrum of G. Show that n is an eigenvalue of $L(G)$ if and only if G^c is disconnected.

(9) Let G be a graph. The normalized Laplacian of G is defined as $N(G) = D^{-\frac{1}{2}}L(G)D^{-\frac{1}{2}}$, where for the degree matrix D with diagonal entries $d_1, \ldots, d_n$, $D^{-\frac{1}{2}}$ is the diagonal matrix with diagonal entries $d_1^{-\frac{1}{2}}, d_2^{-\frac{1}{2}}, \ldots, d_n^{-\frac{1}{2}}$, with the convention that $0^{-\frac{1}{2}} = 0$. Show that (i) $N(G)$ is a positive matrix. (ii) The sum of eigenvalues of $N(G)$ is $n - l$ where l is the number of isolated vertices of G.

(10) Assume that the graph G has no isolated vertices. Using the notation of the previous question: (i) Show that the multiplicity of the eigenvalue 0 of the adjacency matrix $A(G)$ is same as the multiplicity of 1 of $N(G)$. (ii) The multiplicity of the eigenvalue 0 of $N(G)$ is the number of connected components of G.