Indian Statistical Institute, Bangalore
B. Math.
Third Year, First Semester
Analysis on Graphs

Home Assignment III — Due Date : November 17, 2023
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Let A be the adjacency matrix of a connected graph G. Let λ_1 be the largest eigenvalue of A. Take $d_{max} = \max\{d_i : i \in V(G)\}$.

(1) In general $\lambda_1 \leq d_{max}$. Show that $\lambda_1 = d_{max}$ if and only if G is a λ_1- regular graph. (Hint: Make use of Perron-Frobenius theory).

(2) (Vertex coloring) A k-coloring of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$. A k-coloring is said to be a proper coloring if $i \sim j$ implies $f(i) \neq f(j)$ (adjacent vertices are colored differently). The chromatic number $\chi(G)$ of the graph is the smallest natural number k, such that G admits proper k-coloring. The clique number $\omega(G)$ is the size of the largest clique (complete graph) contained in G as a subgraph. Show that (i) $\omega(G) \leq \chi(G) \leq d_{max} + 1$; (ii) $\omega(G) - 1 \leq \lambda_1$. (Here (i) needs just elementary graph theory and for (ii) you may use interlacing inequality).

(3) (Wilf's theorem). Show that $\chi(G) \leq \lfloor \lambda_1 \rfloor + 1$, where $\lfloor \lambda_1 \rfloor$ denotes the integer part of λ_1. (Hint: Use induction and interlacing inequality). Give one example to show that in general $\lfloor \lambda \rfloor < d_{max}$ is possible. In other words this provides a better upper bound for $\chi(G)$ compared to 2(i). (Hint: Try some graphs with small n).

(4) If a graph is strongly regular show that its complement is also strongly regular.

(5) (Hamming graphs) Let Ω be a finite set with at least two elements and let $d \geq 2$ be a natural number. The Hamming graph $H(d, \Omega)$ is defined as follows. The vertex set is the Cartesian product $\Omega \times \Omega \times \cdots \times \Omega$ (d-times). The Hamming distance between two d-tuples $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_1, b_2, \ldots, b_d)$ is the metric D defined by

$$D(a, b) = \#\{j : a_j \neq b_j\}.$$

Two distinct vertices a, b form an edge if $D(a, b) = 1$. Show that $H(d, \Omega)$ is strongly regular if $d = 2$.

(6) Suppose $\mathcal{D}$ is the distance matrix of a graph G. Then $\sum_{1 \leq i < j \leq n} \mathcal{D}_{ij}$ is known as the Wiener index of G. You may read about the importance of this concept in the web. Find examples of graphs whose Weiner indices are 3,4,6,7. Show that the Weiner index can never be equal to 2 or 5. [It is a theorem that the Weiner index can be any natural number different from 2 and 5.]