Assignment-2

1. Let X be a tangential vector field on a param. hypersurface M, $v \in T_pM$. Define $D_v X := \partial_v X - \langle \partial_v X, N(p) \rangle N(p)$; where N is the unit normal field of M. Define, for X, Y tangential fields, $(D_X Y)(p) := D_{X(p)} Y$, $p \in M$. Recall, $[X, Y](p) := \partial_{X(p)} Y - \partial_{Y(p)} X$,
$(\partial_X Y)(p) := \partial_{X(p)} Y - \partial_{Y(p)} X$.

(i) Prove that, for X, Y smooth, tangential,
$$[X, Y] = D_X Y - D_Y X$$

(ii) For $\sigma : \Omega \to \mathbb{R}^n \equiv M$, Prove, for Z tangential
$$\left[\left(D_{\sigma_i} D_{\sigma_j} - D_{\sigma_j} D_{\sigma_i}\right)(Z)\right](p) = \langle L_p(\sigma_j(p)), Z(p) \rangle L_p(\sigma_i(p)) - \langle L_p(\sigma_i(p)), Z(p) \rangle L_p(\sigma_j(p))$$
$\forall p \in M$.

2. For M etc as in 1, $x, y, z \in T_pM$, define
$R(x, y, z) := \langle L_p(y), z \rangle L_p(x) - \langle L_p(x), z \rangle L_p(y)$;
For X, Y, Z tangential on M, $R(X, Y, Z)(p) := R(X(p), Y(p), Z(p))$ and, for W tangential, $R(X, Y, Z, W) := \langle R(X, Y, Z), W \rangle$.

(i) Prove that $R(X, Y, Z) = D_X D_Y - D_Y D_X - D_{[X, Y]}$

(ii) Let $n \geqslant 3$ and $\Pi \subseteq T_pM$ be a 2-dim'l Subspace.

(a) Show that the real number $R(\Pi) = \langle R(e_1, e_2, e_2), e_1 \rangle$, for $\{e_1, e_2\}$ an orthonormal basis of Π, is independent of the choice of the orthonormal basis.

(b) When $n=3$, $R(\Pi) = K(p) =$ Gaussian curvature of M at p.

<u>Remarks</u>: ① The function $R(x,y,z): T_pM \times T_pM \times T_pM \to T_pM$ is called the <u>Riemann tensor</u> of M.

② $R(x,y,z)$ enjoys certain symmetry properties, eg: $R(x,y,z) + R(y,z,x) + R(z,x,y) = 0$, which we will discuss later.

③ The tensors $R(X,Y,Z)$ & $R(X,Y,Z,W)$ are quite abstract and complex in their computations. The above problems relate these to the curvature in low dimensions. Later we shall use these to define curvature of 'connections' in general.