Homework-2

1. Find the Gaussian curvature of

(i) $\sigma(\theta,\varphi) = a(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ (sphere)

(ii) $\sigma(t,\theta) = (\cos\theta, \sin\theta, t)$ (right circular cylinder)

(iii) $\sigma(t,\theta) = (t\cos\theta, t\sin\theta, \theta)$ (helicoid)

2. Let M be a parameterized surface and X a vector field along M (i) give an example to show that, even if X is tangential, for $v \in T_pM$, $\partial_v X$ may fail to be a tangent vector.

(ii) For X tangential & $v \in T_pM$, set

$$D_v X := \partial_v X - (\langle \partial_v X, N(p)\rangle) N(p),$$

where N is the unit normal field of M.

Prove that $D_v(X+Y) = D_v(X) + D_v(Y)$,

$D_v(f.X) = (\nabla_v f)X(p) + f(p)D_v X$,

$\partial_v(\langle X, Y\rangle) = \langle D_v X, Y(p)\rangle + \langle X(p), D_v(Y)\rangle$.

$\forall$ smooth tangential vector fields X, Y & smooth f. We call $D_v X$ the covariant derivative of X with respect to v, $D_v X \in T_pM$.

3. Let M be a param. hypersurface in $\mathbb{R}^n$, N its unit normal, X, Y tangential vector fields on M. For $p \in M$, prove that $\rightsquigarrow$ (PTO)

$$\left\langle \partial_{X(p)} Y, N(p) \right\rangle = \left\langle \partial_{Y(p)} X, N(p) \right\rangle . \; (\forall p)$$

<u>Define</u> the vector field $[X,Y]$ along M by $[X,Y](p) = \partial_{X(p)} Y - \partial_{Y(p)} X$. Prove that, for X, Y tangential to M, $[X,Y]$ is tangential.

4. Let X be a smooth vector field on $\mathbb{R}^n$, $\sigma : \Omega \to \mathbb{R}^n$ a param. hypersurface, $v \in T(\Omega)$. Prove that $\partial_v (X \circ \sigma) = \partial_{D\sigma(v)} X$ where $D\sigma$ is the derivative of σ,
$(D\sigma)(p) : T_p \Omega \to T_{\sigma(p)} \mathbb{R}^n$, $p \in \Omega$.

5. For X, Y, Z vector fields along a hypersurface M, prove the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

The "bracket" $[X,Y]$ of two vector fields is called the Lie bracket (pronounce as Lee bracket!) of X & Y.

6. For tangential vector fields X, Y on M, define $\partial_X Y$ by $(\partial_X Y)(p) := \partial_{X(p)} Y \in T_p M$. Compute $\partial_{[X,Y]} - [\partial_X, \partial_Y]$ for $\mathbb{R}^n$.