Homework - 3

1. Let S be the Cylinder $x_1^2 + x_2^2 = r^2$, $r > 0$ $\subseteq \mathbb{R}^3$ (in its parametric form). Show that $\alpha : I \to S$ is a geodesic of S $\Leftrightarrow$ $\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$ for suitable $a, b, c, d \in \mathbb{R}$.

2. A param. curve α on $S \equiv S^n = \{x_1^2 + \dots + x_{n+1}^2 = 1\} \subseteq \mathbb{R}^{n+1}$ (in its param. form) is a geodesic $\Leftrightarrow$
$$\alpha(t) = (\cos at)\, e_1 + (\sin at)\, e_2$$ for an orthonormal pair $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$.

3. Let S be a param hypersurface $\subseteq \mathbb{R}^n$, $p \in S$, $v \in T_p S$ and $\alpha : I \to S$ a maximal geodesic in S with initial velocity v. Show that the maximal geodesic β in S with initial velocity cv, $c \in \mathbb{R}$, is given by $\beta(t) = \alpha(ct)$.

*4. Let S be an n-plane $a_1 x_1 + \dots + a_{n+1} x_{n+1} = b$ in $\mathbb{R}^{n+1}$, $P, Q \in S$ and let $v = (P, v) \in T_P S$. Show that if α is <u>any</u> param. curve in S from P to Q, then $P_\alpha(v) = (Q, v)$. Conclude from this that in an n-plane, parallel transport is path independent.

*5. Let $\alpha : [0, \pi] \to S^2$ be the half great circle in S^2 running from the north pole $P = (0,0,1)$ to the south pole $Q = (0,0,-1)$, defined by $\alpha(t) = (\sin t, 0, \cos t)$. Show that, for $v = (P, (v_1, v_2, 0)) \in T_P S^2$, $P_\alpha(v) = (Q, (-v_1, v_2, 0))$.

(Hint: Check this first for $v = (p, (1,0,0))$ & for $v = (p,(0,1,0))$; then use linearity of P_α).

*6. Let S be a param n-hypersurface $\subseteq \mathbb{R}^{n+1}$; $p \in S$; let $G_p = \{\phi : T_pS \to T_pS,\ \phi \text{ bijective linear}\}$
$= GL(T_pS)$.

Let $H_p := \{T \in G_p : T = P_\alpha$ for some piecewise smooth $\alpha : [a,b] \to S$ with $\alpha(a) = \alpha(b) = p\}$.

Show that H_p is a subgroup of G_p by showing

(i) for each pair of piecewise smooth α, β in $S : p \to ~~p~~ p$, $\exists$ a piecewise smooth curve $\gamma : p \to p$ such that $P_\gamma = P_\beta \circ P_\alpha$,

(ii) for each α in $S : p \to p$, $\exists\ \beta$ in S $p \to p$ such that $P_\beta = P_\alpha^{-1}$.

(H_p is called the holonomy group of S at p)

[We'll discuss starred problems after the next class]