Homework-2 (Post Midsem)

1. Let M, N be smooth manifolds, $f : M \to N$ smooth. Assume M is compact & N is connected. If f is injective and $Df(p) : T_pM \to T_{f(p)}N$ an isomorphism $\forall p \in M$, prove that f is a diffeomorphism. 2

2. Does there exist a smooth vector field on S^1 which is not left invariant? 2

3. Recall the Lie group structure on S^3, via the Hamilton's quaternion $\mathbb{R}$-algebra $\mathbb{H}$ thought of as the set of all norm 1 elements in $\mathbb{H}$. Consider the vector field X on S^3 given by
$$X(p) = \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}\right) + \left(-x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}\right).$$
Is X left-invariant on S^3?

4. Let M be smooth manifold, $p \in M$ & $v \in T_pM$. Prove that $\exists\, X \in \mathfrak{X}(M)$ with $X(p) = v$. 2

5. Consider the exponential map $\exp : M_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $\exp(X) = I + X + \frac{X^2}{2} + \cdots$. Is $\exp : M_2(\mathbb{R}) \to GL_2(\mathbb{R})$ injective? Surjective?

6. Let M be a compact smooth manifold of dim n and $f : M \to \mathbb{R}^n$ smooth. Prove that $Df(p)$ cannot be injective $\forall p \in M$.

7. Let M be a smooth manifold, $X \in \mathfrak{X}(M)$. Let γ be an integral curve of X and suppose $\dot{\gamma}(t_0) = 0$ for some t_0. Prove γ must be constant.

8. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\rho: G \to GL(V)$ be a representation with ρ smooth. Let $v \in V$ be such that $\rho(g)v = v \ \forall g \in G$, i.e. $v \in V^G = \{x \in V \mid \rho(g)x = x \ \forall g \in G\}$. Prove that $d\rho(X)v = 0 \ \forall X \in \mathfrak{g}$. 2

9. Let M be a smooth manifold, $X, Y \in \mathfrak{X}(M)$. Let $\{\varphi_s^X\}$ & $\{\varphi_t^Y\}$ be the associated local 1-parameter families of local diffeomorphisms. Prove that

$$[X, Y] = 0 \Rightarrow \varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X \quad \forall s, t.$$

10. Let V be a finite dimensional $\mathbb{R}$-vector space, $B: V \times V \to \mathbb{R}$ be non-degenerate ($B(x,y) = 0 \ \forall y \Rightarrow x = 0$).

Let $G = \{g \in GL(V) \mid B(gx, gy) = B(x,y) \ \forall x, y \in V\}$.

Prove that G is a Lie subgroup of $GL(V)$ and

$$\mathrm{Lie}(G) = \{X \in \mathfrak{gl}(V) \mid B(Xv, w) + B(v, Xw) = 0 \ \forall v, w \in V\}.$$

11. Let G, H be Lie groups with $\mathfrak{g}, \mathfrak{h}$ their Lie algebras. Let $\varphi: G \to H$ be a Lie group homomorphism. Show that $\ker \varphi$ is a (normal) Lie subgroup of G & $\mathrm{Lie}(\ker\varphi) = \ker(D\varphi(e))$, e = identity element of G.

12. Using (10), (11), compute Lie algebras of $SO(n)$, SL_n, SP_n.

13. Let $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$ be the map $(a,b,c) \mapsto (a^l, b^m, c^n)$, $l,m,n \geq 1$. Let $\omega = dx \wedge dy \wedge dz$. Compute the pullback $\varphi^*(\omega)$.

14. For $\alpha = x^2\,dx + xy\,dy \in D^1(\mathbb{R}^2)$, find $d\alpha$.

15. Let $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ be smooth & $x_1, \ldots, x_n$ the usual coordinates. Compute $\varphi^*(dx_1 \wedge \cdots \wedge dx_n)$

16. The form $dx_1 \wedge \cdots \wedge dx_n$ is called the volume form of $\mathbb{R}^n$. Let $\overset{\shortparallel}{\omega}$ & $x \in \mathbb{R}^n$. Compute $f^*(\omega)(x)$ $\in \Lambda^n(T_x\mathbb{R}^n)^* \equiv \Lambda^n T_x\mathbb{R}^n \equiv \Lambda\,\mathbb{R}^n$.

17. Prove that Lie groups are orientable

18. Let $\varphi_1, \ldots, \varphi_p$ be 1-forms on a manifold & $X_1, \ldots, X_p \in \mathfrak{X}(M)$. Show that

$$(\varphi_1 \wedge \cdots \wedge \varphi_p)(X_1, \ldots, X_p) = \sum_{\sigma \in S_p} \operatorname{sign}(\sigma)\, \varphi_1(X_{\sigma(1)}) \cdots \varphi_n(X_{\sigma(n)})$$

19. Let ω be a differential form on $\mathbb{R}^n$. Is then $\omega \wedge \omega = 0$?

20. Let $\varphi: \mathbb{R} \to \mathbb{R}^2 \setminus (0,0)$ be $\varphi(\theta) = (\cos\theta, \sin\theta)$ & ω be the differential 1-form on $\mathbb{R}^2 \setminus (0,0)$,

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$ Compute $\varphi^*(\omega) \in D^1(\mathbb{R})$.

Show that ω is not exact, but is closed.

21. Let $G \subseteq GL_n(\mathbb{R})$ be a Lie group, $\mu: G \times G \to G$ & $\iota: G \to G$ be the product & inversion maps resp. Prove that $D\mu(e,e)(X,Y) = X+Y$ & $D\iota(e)(X) = -X$ for $X, Y \in \operatorname{Lie}(G)$, $e =$ Identity element of G.