Assignment-1

1. Let $U \subseteq \mathbb{R}^n$ be open, $f: U \to \mathbb{R}^n$ be a continuously differentiable injective function such that $Df(x)$ is invertible $\forall\, x \in U$. Show that $f(U)$ is open and $f^{-1}: f(U) \to U$ is differentiable. Prove that $f(V)$ is open for any open set $V \subseteq U$. (1)

(continuously differentiable $\equiv$ diff. at all points of U and $\partial f_i / \partial x_j$ are all continuous.)

2. Let $f: \mathbb{R} \to \mathbb{R}$ be a C^r-local diffeomorphism at each point, $r \geq 1$; then prove that f is a global C^r-diffeomorphism. (1)

—x—

Assignment - 2

1. Prove that a map $X : M \to \bigcup_p T_pM$ with $X(p) \in T_pM$ $\forall\, p \in M$, defines a smooth vector field on a smooth manifold M if and only if $\forall\, f \in C^\infty(M)$, the map $X_f : M \to \mathbb{R}$ given by $X_f(p) := \cancel{X_p(f)}\; X(p)(f)$ is smooth. (1)

2. Let M be a smooth manifold, $p \in M$ and $v_0 \in T_p(M)$ be fixed. Prove that $\exists$ a smooth vector field X on M such that $\forall\, q \in M$, $X(q)$ is defined and $X(p) = v_0$; hence any tangent vector can be 'extended' to a vector field on M. (1).

Assignment-3

1. Let M be a compact smooth manifold of dimension n and $f: M \to \mathbb{R}^n$ a smooth map. Prove that $Df(p)$ cannot be 1-1 for <u>all</u> $p \in M$.

(1)

2. Let M be a smooth manifold, $X \in \mathfrak{X}(M)$. Let $\gamma(t)$ be an integral curve of X. Suppose that $\dot{\gamma}(t_0)=0$ for some t_0. Prove that γ must be a constant map, i.e. the image of γ is a single point.

(1)

Assignment-4

1. Prove that $U(n)$ and $SU(n)$ are path connected.

(1)

2. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ and $\rho : G \to GL(V)$ a representation (smooth) of G. Then $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$. Suppose $v \in V$ is such that $\rho(g).v = v \ \forall\, g \in G$. Prove that $d\rho(X).v = 0 \ \forall\, X \in \mathfrak{g}$.

(1).

Assignment-5

1. Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the map $\varphi((a,b,c)) = (a^l, b^m, c^n)$, here $l, m, n \geq 1$ are fixed positive integers. Compute the pullback $\varphi^*(\omega)$, where $\omega = dx \wedge dy \wedge dz$ on $\mathbb{R}^3$, x, y, z denote the usual coordinates.

(1)

2. Let $\alpha = x^2 dx + xy\, dy \in D^1(\mathbb{R}^2)$.
Compute $d\alpha$.

(1).