Homework-1

1. Let $f : U \to \mathbb{R}^m$ be differentiable at $a \in U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. Prove that for $x \in \mathbb{R}^n$, $Df(a)(x) = \lim_{t \to 0} [f(a+tx) - f(a)]$-

2. Identify $M_n(\mathbb{R})$ = vector space of all $n \times n$ matrices with entries in $\mathbb{R}$, with $\mathbb{R}^{n^2}$. Then $GL_n(\mathbb{R})$ = group of $n \times n$ invertible matrices in $M_n(\mathbb{R})$, is an open subset of $M_n(\mathbb{R})$. Prove that $i : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $i(x) = x^{-1}$ is differentiable at all points of $GL_n(\mathbb{R})$.

3. Let $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $f : U \to \mathbb{R}^n$ be C^r $(r \geq 1)$ and assume f is a C^1-diffeomorphism (onto its image). Prove that f is a C^r-diffeomorphism.

4. $U, V \subseteq \mathbb{R}^n$ be open, $f : U \to V$ a smooth bijection (smooth $\equiv C^\infty$). Then f is a smooth diffeomorphism if and only if f is a C^1-local diffeomorphism at all $p \in U$.

5. Given an example to show that $C^{r+1} \subsetneq C^r$ in general.

6. $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x}{2} + x^2 \sin \frac{1}{x}$, $x \neq 0$, $f(0) = 0$, has non-vanishing derivative at $x = 0$, but f is NOT a diffeomorphism in any neighbourhood of $x = 0$.