Homework - 3

1. Let M be a smooth manifold of dimension m. Let $\mathcal{T}_p$ for $p \in M$ be defined as the set of all triples (U, φ, u) where (U,φ) is a coordinate chart around p, and $u \in \mathbb{R}^m$. Define a relation $\sim$ on $\mathcal{T}_p$ by
$(U,\varphi,u) \sim (V,\psi,v) \iff D(\varphi\circ\psi^{-1})(\psi(p))(v) = u$.
   - Show that $\sim$ is an equiv. relation on $\mathcal{T}_p$.
   - Define a (natural?) bijection $\mathcal{T}_p/\sim \;\to T_pM$.

(This gives a way to define tangent vectors as we do for embedded manifolds.)

2. Let $C^\infty(p)$, for $p \in M$, be the $\mathbb{R}$-algebra of $\mathbb{R}$-valued functions f, smooth and defined in a neighbourhood U_f of p in M. Define the equivalence (?) on $C^\infty(p)$ by $(f,U_f) \sim (g,U_g)$ $\iff f = g$ on $U_f \cap U_g$. The equivalence classes are called <u>germs of smooth functions at p</u>.
   - Let $\mathcal{C}_p = C^\infty(p)/\sim$. Then $\mathcal{C}_p$ is an $\mathbb{R}$-algebra in a natural way. Let $m_{\mathcal{C}_p}$ = set of germs that vanish at p. Show: $m_{\mathcal{C}_p}$ is a max ideal of $\mathcal{C}_p$. Prove $\exists$ a natural isomorphism of $\mathbb{R}$-vector spaces $\mathrm{Hom}_{\mathbb{R}}\left(m_{\mathcal{C}_p}/m_{\mathcal{C}_p}^2, \mathbb{R}\right) \xrightarrow{\sim} T_pM$.

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