~~To be submitted~~

Homework-5

1. Does there exist a smooth vector field X on S^1 which is not left-invariant? Explain.

2. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}\cdot ij$ be the real quaternions, $i^2 = j^2 = -1$, $ij = -ji$. The norm of a quaternion $x = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $k = ij$, is given by $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$. Prove that (i) the norm $n(x)$ equals $x \cdot \bar{x}$ where $\bar{x} := \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$.

~~To be submitted~~ (ii) $n(xy) = n(x)n(y)$, $\forall x, y \in \mathbb{H}$.

3. The norm 1 quaternions in $\mathbb{H}$ form the unit sphere $S^3 = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4 \mid \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\}$.

   Prove that S^3 is a Lie group under the quaternionic multiplication defined in (1) by the product of basis elements $\{1, i, j, k = ij\}$.

4. The nowhere vanishing vector field on S^3: $X(p) = \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}\right) + \left(-x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}\right)$; is X left invariant on S^3? Explain.