Homework-1

1. Let $f: U \to \mathbb{R}^m$ be differentiable at $a \in U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. Prove that for $x \in \mathbb{R}^n$, $Df(a)(x) = \lim_{t \to 0} [f(a+tx) - f(a)]$-

2. Identify $M_n(\mathbb{R})$ = vector space of all $n \times n$ matrices with entries in $\mathbb{R}$, with $\mathbb{R}^{n^2}$. Then $GL_n(\mathbb{R})$ = group of $n \times n$ invertible matrices in $M_n(\mathbb{R})$, is an open subset of $M_n(\mathbb{R})$. Prove that $i: GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $i(x) = x^{-1}$ is differentiable at all points of $GL_n(\mathbb{R})$.

3. Let $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $f: U \to \mathbb{R}^n$ be C^r $(r \geq 1)$ and assume f is a C^1-diffeomorphism (onto its image). Prove that f is a C^r-diffeomorphism.

4. $U, V \subseteq \mathbb{R}^n$ be open, $f: U \to V$ a smooth bijection (smooth $\equiv C^\infty$). Then f is a smooth diffeomorphism if and only if f is a C^1-local diffeomorphism at all $p \in U$.

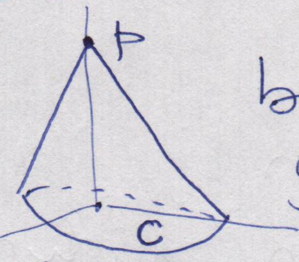
5. Given an example to show that $C^{r+1} \subsetneq C^r$ in general.

6. $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x}{2} + x^2 \sin \frac{1}{x}$, $x \neq 0$, $f(0) = 0$, has non-vanishing derivative at $x = 0$, but f is NOT a diffeomorphism in any neighbourhood of $x = 0$.

Homework-2

1. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ with the subspace topology, $N = (0,1) \in \mathbb{R}^n \times \mathbb{R}$ & $S = (0,-1) \in \mathbb{R}^n \times \mathbb{R}$ be the north & south poles respectively. Let $U_N = S^n \setminus \{N\}$ & $U_S = S^n \setminus \{S\}$ and $\varphi_N : U_N \to \mathbb{R}^n$, $\varphi_S : U_S \to \mathbb{R}^n$ be given by $\varphi_N((a_1, \dots, a_{n+1})) = \frac{1}{1-a_{n+1}}(a_1, \dots, a_n)$ and $\varphi_S((a_1, \dots, a_{n+1})) = \frac{1}{1+a_{n+1}}(a_1, \dots, a_n)$. Describe the transition maps $\varphi_S^{-1} \circ \varphi_N$, $\varphi_N \circ \varphi_S^{-1}$ and Check the smoothness of S^n with the atlas $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$.

2. Let $\mathcal{A} = \{(U_i, \varphi_i), 1 \leq i \leq n+1\}$ be the atlas of $\mathbb{R}P^n$ as defined in Lecture-6. Describe the transition maps for $\mathcal{A}$ and Check smoothness of $(\mathbb{R}P^n, \mathcal{A})$.

3. Let $X =$ 

be the cone in $\mathbb{R}^3$ given suitably, with the boundary circle C removed. Can we give an atlas for X so that it is a smooth manifold? Explain.

Homework - 3

1. Let M be a smooth manifold of dimension m. Let τ_p for $p \in M$ be defined as the set of all triples (U, φ, u) where (U, φ) is a coordinate chart around p, and $u \in \mathbb{R}^m$. Define a relation $\sim$ on τ_p by
$(U, \varphi, u) \sim (V, \psi, v) \iff D(\varphi \circ \psi^{-1})(\psi(p))(v) = u$.
   - Show that $\sim$ is an equiv. relation on τ_p.
   - Define a (natural?) bijection $\tau_p / \sim \; \to T_pM$.

(This gives a way to define tangent vectors as we do for embedded manifolds.)

2. Let $C^\infty(p)$, for $p \in M$, be the $\mathbb{R}$-algebra of $\mathbb{R}$-valued functions f, smooth and defined in a neighbourhood U_f of p in M. Define the equivalence (?) on $C^\infty(p)$ by $(f, U_f) \sim (g, U_g) \iff f = g$ on $U_f \cap U_g$. The equivalence classes are called <u>germs of smooth functions at p</u>.
   - Let $\mathcal{C}_p = C^\infty(p)/\sim$. Then $\mathcal{C}_p$ is an $\mathbb{R}$-algebra in a natural way. Let $\mathfrak{m}_{\mathcal{C}_p}$ = set of germs that vanish at p. Show: $\mathfrak{m}_{\mathcal{C}_p}$ is a max ideal of $\mathcal{C}_p$. Prove $\exists$ a natural isomorphism of $\mathbb{R}$-vector spaces $\mathrm{Hom}_{\mathbb{R}}\left(\mathfrak{m}_{\mathcal{C}_p} / \mathfrak{m}_{\mathcal{C}_p}^2, \mathbb{R}\right) \xrightarrow{\sim} T_pM$.

Homework-4

~~To be submitted~~

1. Let M be a smooth manifold, $f, g \in C^\infty(M)$ be such that on some open subset $V \subseteq M$, $f|_V = g|_V$ (i.e. $f(x) = g(x)\ \forall x \in V$). Prove that $X(f) = X(g)$.

2. Recall: We have charts on $\mathbb{RP}^2$ given by
$$[(x,y,z)] \longmapsto (u_1, u_2) = (x/z,\ y/z) \text{ on } U_3 = \{z \neq 0\},$$
$$[(x,y,z)] \longmapsto (v_1, v_2) = (x/y,\ z/y) \text{ on } U_2 = \{y \neq 0\}$$
$$\& \quad [(x,y,z)] \longmapsto (\omega_1, \omega_2) = (y/x,\ z/x) \text{ on } U_1 = \{x \neq 0\}.$$
Prove that $\exists$ a smooth vecter field on $\mathbb{RP}^2$ which, on U_1, has the expression
$$\omega_1 \frac{\partial}{\partial \omega_1} - \omega_2 \frac{\partial}{\partial \omega_2}.$$
~~To be submitted~~
Write the expression for this vecter field in the other two charts.

3. M, N be smooth, $f: M \to N$ smooth. Suppose M is compact & N connected. If f is injective and $Df(p)$ is an isomorphism $: T_pM \to T_pN$ for all p, prove that f is a diffeomorphism.

~~To be submitted~~

Homework-5

1. Does there exist a smooth vector field X on S^1 which is not left-invariant? Explain.

2. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}\cdot ij$ be the real quaternions, $i^2 = j^2 = -1$, $ij = -ji$. The norm of a quaternion $x = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, $k = ij$, is given by $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$. Prove that (i) the norm $n(x)$ equals $x \cdot \bar{x}$ where $\bar{x} := \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$.

~~To be submitted~~ (ii) $n(xy) = n(x)n(y)$, $\forall x, y \in \mathbb{H}$.

3. The norm 1 quaternions in $\mathbb{H}$ form the unit sphere $S^3 = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^4 \mid \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\}$.

   Prove that S^3 is a Lie group under the quaternionic multiplication defined in (1) by the product of basis elements $\{1, i, j, k = ij\}$.

4. The nowhere vanishing vector field on S^3: $X(p) = \left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}\right) + \left(-x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}\right)$; is X left invariant on S^3? Explain.

Homework-6

1. Let G be a Lie group. A one-parameter subgroup of G is a Lie group homomorphism $: \mathbb{R} \to G$ (definition).
Let $\varphi : \mathbb{R} \to G$ be a 1-parameter subgroup of G. Show that $\exists\, X \in \mathfrak{g}$ such that $\varphi(t) = \mathrm{Exp}_{tX}(e)$.
Conversely, let $X \in \mathfrak{g}$ and $\{\alpha_t\}$ be the associated 1-parameter group of diffeomorphisms. Define $\varphi(t) := \alpha_t(e)$. Show that $\varphi : \mathbb{R} \to G$ is a 1-parameter subgroup of G. and $\mathrm{Exp}_{tX} = R_{\varphi(t)}$, the right translation map $R_{\varphi(t)} : G \to G$.

2. Let M be a manifold, $X, Y \in \mathfrak{X}(M)$ and $\{\varphi^X_s\}$, $\{\varphi^Y_t\}$ be the associated 1-parameter (local) families of local diffeomorphisms.
Prove that $[X, Y] = 0 \Rightarrow \varphi^X_s \circ \varphi^Y_t = \varphi^Y_t \circ \varphi^X_t$ in the appropriate domains of definition for s, t in the appropriate domains of definition.

Homework-7

1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1. Compute the pullback $f^*(dx_1 \wedge \cdots \wedge dx_n)$; $x_1, \ldots, x_n$ denote the usual coordinates on $\mathbb{R}^n$.

2. The n-form $dx_1 \wedge \cdots \wedge dy_n$ on $\mathbb{R}^n$ is called the <u>volume</u> <u>form</u> on $\mathbb{R}^n$. Let $\omega = dx_1 \wedge \cdots \wedge dx_n$ & $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth, $x \in \mathbb{R}^n$. Compute
$$f^*(\omega)(x) \in \Lambda^n (T_x\mathbb{R}^n)^* \equiv \Lambda^n T_x \mathbb{R}^n \equiv \Lambda^n \mathbb{R}^n.$$

3. Let g be an orientation preserving diffeom. $: U \xrightarrow{\sim} V$, both open subsets of $\mathbb{R}^n$. Let α be a compactly supported n-form on V. Show that
$$\int_U g^*(\alpha) = \int_V \alpha .$$

4. What can you say about the groups $H^r_{dR}(\mathbb{R}^n)$ for $r > n$?

5. Generalize your answer for 4!

More problems may be posted next week!

Homework-8

1. Let $F := (F_1, \cdots, F_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a smooth function. Let $\{x_i\}$ be the coordinates on $\mathbb{R}^n$ & $\{y_j\}$ the coordinates in $\mathbb{R}^m$. Compute $F^* dy_i$, $1 \leq i \leq m$.
2. Let $f, g \in C^\infty(M)$. Prove that $d(fg) = f\,dg + g\,df$.
3. Prove that Lie groups are orientable.
4. Let $\dim M = m$, then $H^r_{dR}(M) = 0$, $r \geq m+1$.
5. Let $\varphi_1, \cdots, \varphi_p$ be 1-forms, $X_1, \cdots, X_p$ vector fields then $(\varphi_1 \wedge \cdots \wedge \varphi_p)(X_1, \cdots, X_p) = \sum_{\sigma \in S_p} \operatorname{sign}(\sigma)\, \varphi_1(X_{\sigma(1)}) \cdots \varphi_n(X_{\sigma(n)})$
6. Let ω be a differential form on $\mathbb{R}^n$. Is it true that $\omega \wedge \omega = 0$?
7. Consider $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$, $\varphi(\theta) = (\cos\theta, \sin\theta)$. Let ω be the differential 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$,
$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$
Compute the pull back $\varphi^* \omega \in D^1(\mathbb{R})$.
8. Consider $\mathbb{R}^2$ with polar coordinates (r, θ) and $\omega = (r \sin\theta)\,dr \in D^1(\mathbb{R}^2)$. Compute $d\omega$.
9. Let α, β & γ be differential forms with $d\alpha = 0 = d\beta = d\gamma$. What can you say about $d(\alpha \wedge \beta \wedge \gamma)$?
10. Let ω be as in (7). Show that ω is <u>not</u> exact.