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Vector fields :- Let M be a smooth manifold. By a smooth vector field on M, we mean a map $X : M \longrightarrow T(M) = \bigcup_p T_pM$ such that

- $X(p) \in T_pM \ \forall \ p \in M$
- For any coordinate chart $U \ni p$ with coordinates $x_1, \dots, x_m$; Write $X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i}\Big|_q$, $q \in U$.

We demand that f_i are smooth with respect to any local chart in the smooth atlas of M.

Remark: When we understand the tangent bundle $T(M)$ is a smooth manifold, we'll see that $X : M \to T(M)$ with $X(p) \in T_pM$, is smooth if it is so as a map of manifolds.

Example :- $U \subseteq \mathbb{R}^n$ be open, $X_i(p) = \frac{\partial}{\partial x_i}\Big|_p$ for $p \in U$. Then X_i is a smooth vector field on U, $1 \leq i \leq n$. For $U = \mathbb{R}^n$ and $p \in \mathbb{R}^n$, $X(p) = \sum f_i(p) \frac{\partial}{\partial x_i}\Big|_p$, $f_i \in C^\infty(\mathbb{R}^n)$ is a smooth vector field.

- Any smooth vect. field on $\mathbb{R}^n$ is as above.

<u>Remarks</u>: 1. As we discussed tangent vectors to M at p as derivations on $C^\infty(p)$, we'll see that we can regard smooth vector fields on M as derivations : $C^\infty(M) \to C^\infty(M)$.

2. Let $k \geq 1$ & $n = 2k-1$, $M = S^n \subset \mathbb{R}^{n+1}$, the n-sphere, n odd. Then we have seen that $T_p S^n \subsetneq T_p \mathbb{R}^{n+1}$. For $p \in S^n$, let

$$X(p) = \left[\left(-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}\right) + \left(-x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}\right) + \cdots + \left(-x_{2k} \frac{\partial}{\partial x_{2k-1}} + x_{2k-1} \frac{\partial}{\partial x_{2k}}\right)\right]\Bigg|_p$$

<u>Exercise</u>:
- $X(p) \in T_p(S^n)$ $\forall p \in S^n$, so X is a smooth vector field on S^n.
- $X(p) \neq 0$ $\forall p \in S^n$.
- For S^n, n even, every (continuous) vector field X admits a zero, i.e. $\exists p \in S^n$, $X(p) = 0$. (see Kum.)

<u>Vector fields as derivations</u>:- Let M be smooth, $C^\infty(M)$ = $\mathbb{R}$-algebra of smooth func's $: M \to \mathbb{R}$. Let $\mathfrak{X}(M) :=$ set of all smooth vector fields on M.

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- We have natural operations on $\mathcal{X}(M)$:

$(X+Y)(p) = X(p)+Y(p) \in T_pM$, $p \in M$;

For $\alpha \in \mathbb{R}$, $X \in \mathcal{X}(M)$, $(\alpha\cdot X)(p) = \alpha\cdot X(p) \in T_p(M)$.

Then • $X+Y$, $\alpha X \in \mathcal{X}(M)$.

This makes $\mathcal{X}(M)$ into an $\mathbb{R}$-vector space.

<u>$C^\infty(M)$-module structure on $\mathcal{X}(M)$</u>: Let $f \in C^\infty(M)$ and $X \in \mathcal{X}(M)$. We define $f\cdot X$ by $(f\cdot X)(p) := f(p)X(p) \in T_pM$. Since $f \in C^\infty(M)$, it follows that $f\cdot X \in \mathcal{X}(M)$.

Hence $\mathcal{X}(M)$ is a $C^\infty(M)$-module.

<u>Derivations of $C^\infty(M)$</u>: Let $X \in \mathcal{X}(M)$.

Define $\tilde{X} : C^\infty(M) \to C^\infty(M)$ by $\tilde{X}(f)(p) := X(p)(f)$.

Then • $\tilde{X}(f) \in C^\infty(M)$, $\tilde{X}(\alpha f + \beta g) = \alpha \tilde{X}(f) + \beta \tilde{X}(g)$.

• $\tilde{X}(fg) = \tilde{X}(f)\cdot g + f\cdot \tilde{X}(g)$.

→ We'll prove that any derivation of $C^\infty(M)$ comes from a smooth vector field.

<u>Functions (smooth) with compact support</u>:-

• Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0. \end{cases}$

Then $f \in C^\infty(\mathbb{R})$ (but not analytic at 0!). →

107. Let $0 < a < b$ and define $\varphi : \mathbb{R}^n \to \mathbb{R}$ by
$$\varphi(x) = \frac{f(b^2 - \|x\|^2)}{\left[f(b^2 - \|x\|^2) + f(\|x\|^2 - a^2)\right]}.$$

- Then $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(x) = \begin{cases} 1 & \text{if } x \in B(0,a) \\ 0 & \text{if } x \notin B(0,b). \end{cases}$

→ For $\varphi \in C^\infty(U)$, $U \subseteq \mathbb{R}^n$ open, we define

Support of $\varphi = \overline{\{x \in U \mid \varphi(x) \neq 0\}}$.

<u>Prop</u>: $K \subseteq \mathbb{R}^n$ be compact & $U \subseteq \mathbb{R}^n$ be open, $K \subseteq U$. Then $\exists\, f \in C^\infty(\mathbb{R}^n)$, $f \equiv 1$ on K, $f = 0$ on $\mathbb{R}^n \setminus U$.

<u>Prf</u>:- $C = \mathbb{R}^n \setminus U$ is closed, $C \cap K = \phi$.

Let $2\epsilon := d(K, C) = \inf\{d(x,y) \mid x \in K, y \in C\}$.

→ $\epsilon > 0$. Let $x_1, \dots, x_k \in K$ be such that $K \subseteq \bigcup_i B(x_i, \epsilon/2)$. Choose $\psi_j \in C^\infty(\mathbb{R}^n)$, $\psi_j = 1$ on $B(x_j, \epsilon/2)$, $\psi_j(x) = 0$ for $x \notin B(x_j, \epsilon)$. Then
$$\varphi(x) := 1 - \prod_{j=1}^{k} (1 - \psi_j)$$ has the req. property. $\boxtimes$

<u>Exercise1</u>:- Let $U \subseteq \mathbb{R}^n$ open, $f \in C^\infty(U)$, $x_0 \in U$. Then $\exists\, V \subseteq U$ open, $x_0 \in V$ & $g \in C^\infty(\mathbb{R}^n)$ s.t.
$$g(x) = \begin{cases} f(x), & x \in V \\ 0, & x \notin U. \end{cases}$$

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Exercise 2 :- • We construct smooth functions on chart nbds in M by using such functions on (open subsets of) $\mathbb{R}^m$ and pulling them back by the corresp. charts.

• To get smooth maps : $M \to \mathbb{R}$, let (U, φ) be a coord. chart on M, $F : \varphi(U) \to \mathbb{R}$ be smooth with compact support $\subseteq \varphi(U)$. Define $f : M \to \mathbb{R}$ by $f = F \circ \varphi$ on U and $f = 0$ on $M \setminus U$. Then $f \in C^\infty(M)$ has compact support $\subseteq U$.

Theorem :- Let $\delta : C^\infty(M) \to C^\infty(M)$ be derivation. Then $\delta = \tilde{X}$ for a! smooth vector field X on M.

Proof :- 1. Let $X, Y \in \mathfrak{X}(M)$ be such that $\tilde{X} = \tilde{Y}$. Then $X = Y$. Suffices to show : if $\tilde{X}(f) = 0 \ \forall f \in C^\infty(M)$ then $X = 0$.

We'll show $X(p) = 0 \ \forall p \in M$. Let (U, φ) be a chart around p. Let $g \in C^\infty(M)$ be such that $g = 1$ in a nbd of $p \subseteq U$ and $g = 0$ on $M \setminus U$. Let $f := \begin{cases} g x_i & \text{on } U \\ 0 & \text{on } M \setminus U. \end{cases}$, x_i coordinates on U. Then $f \in C^\infty(M)$. By hypothesis $\tilde{X}(f) = 0$.

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But $\underbrace{\tilde{X}(f)(q)}_{\overset{\shortparallel}{0}} = X(q)(f) = X(q)(g x_i)$
$= X(q)(x_i) \ \forall q \in V.$

Hence the local coordinates of $X(q)$ on V are 0. and thus $X(q) = 0 \ \forall q \in V$. In part. $X(p) = 0$.

2. If $D : C^\infty(M) \to C^\infty(M)$ is a derivation and $f = 0$ on an open set $U \subseteq M$, then $Df = 0$ on U. To see this, let $p \in V_p \subset U_p \subset U$, all open, $g \in C^\infty(M)$ be s.t. $g = 1$ on V_p, $g = 0$ on $M \setminus U_p$. Set $h = 1 - g$. Then $f = fh$ on M: if $z \in U_p$, then $f(z) = 0 = h(z) f(z)$
$= (fh)(z)$.

If $z \notin U_p$, $h(z) = (1-g)(z) = 1$. Hence $f(z) = f(z) h(z) = (fh)(z)$.

- $D(f) = D(fh) = f \cdot D(h) + D(f) \cdot h$. Hence $D(f)(p) = \underbrace{f(p)}_{\overset{\shortparallel}{0}} (Dh)(p) + (Df)(p) \underbrace{h(p)}_{\overset{\shortparallel}{0}} = 0$.

3. To define a vector field X, we must give $X(p) \in T_p M \ \forall p \in M$, i.e. a derivation $: C^\infty(p) \to \mathbb{R}$, $\forall p$. Let $f \in C^\infty(p)$ and define $X(p)(f) := \delta(g)(p)$ for any $\rightsquigarrow$

10-10 $g \in C^\infty(M)$, $g = f$ in a nbd of p. Then δ is a derivation $\Rightarrow X(p) \in T_p(M)$.

If $h \in C^\infty(M)$ with $h = f$ in a nbd of p, then $h = g$ in a nbd of p, so $\delta(g-h) = 0$ on that nbd; hence $\delta(g) = \delta(h)$. Thus $X(p)$ is well defined. Thus we have a vector field $X : p \mapsto X(p)$ on M.

To show X is smooth, fix a chart $(U, \varphi) \ni p$. There are $f_i \in C^\infty(M)$ s.t. $f_i = x_i$ on a nbd of p. Then the coefficients of X are given by $X(x_i) = X(f_i)$ on U. Since $X(f_i) = \delta(f_i)$ are smooth, it follows that X is smooth. $\boxtimes$

<u>Remark</u>: We have therefore established a natural isomorphism of vector spaces

$$\mathfrak{X}(M) \longrightarrow \mathrm{Der}(C^\infty(M), C^\infty(M))$$
$$X \longmapsto \tilde{X}.$$

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