Lecture-10

Exercises: 1. Let S be a level set of a smooth map $f: \mathbb{R}^{n+k} \to \mathbb{R}^n$ so that S is a smooth mfld. Prove that the inclusion $\iota: S \to \mathbb{R}^{n+k}$ is an immersion. So $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an immersion.

2. The quotient map $\pi: S^n \to \mathbb{RP}^n$ is an immersion as well as a submersion (See Kum.)

Submanifolds :- • Given a smooth manifold, $N \subseteq M$, we would like N to be a smooth submanifold if the inclusion is smooth. Another point is that for a "smooth submanifold" N, $p \in N$, we would like T_pN to be a subspace of T_pM. Finally, restrictions of smooth maps $: M \to W$ to N should be smooth. These requirements suggest:

Def :- A submanifold of a smooth manifold M is a pair (S, ι), where S is a smooth manifold, $\iota: S \to M$ is a 1-1 immersion.

Examples: 1. Level surface manifolds (Exc 1) are submanifolds of the ambient Euclidean space

2. (U, ι), $U \subseteq M$ open subset & ι = inclusion.

Exercise : $f: M \to N$ be smooth, then f is continuous.

Remarks (see Kum.) 1. If (S, ι) is a submanifold of M, the subspace topology on $\iota(S) \subseteq M$ induces a topology on S, which in general is coarser than the topology S comes with.

2. For $p \in S$, $\exists$ U open $\subseteq S$, $p \in (U, x) \subseteq S$, $q = \iota(p) \in (V, y) \subseteq M$ such that

$$y \circ \iota \circ x^{-1}(x_1, \ldots, x_s) = (y_1, \ldots, y_m, 0, \ldots, 0);$$

i.e. $$y \circ \iota \circ x^{-1}(x(U)) = \{q \in y(V) \mid y_i(q) = 0, \ i \geq m+1\}.$$

Call (U, x), (V, y) as above as adapted charts.

Def^n: A smooth map $\varphi: M \to N$ is an imbedding if φ is 1-1 immersion on M. Call φ, an imbedding, regular if $\varphi: M \to \varphi(M)$ is a homeom.

3. If $\varphi: M \to N$ is a regular imbedding, then the original topology on M & the one induced from N via φ coincide.

4. A submanifold of N is a pair (M, ι) with ι an imbedding. A regular submanifold is (S, ι) with ι a regular imbedding.

Regular values of smooth maps :- Let $f: M \to N$ be a smooth map of manifolds. $\rightarrow$

We call a point $q \in N$ a <u>regular value</u> of f if f is a <u>submersion</u> at all $p \in f^{-1}(q)$.

We then have:

- <u>Propn 1</u>:- Let $q \in N$ be a regular value of $f: M \to N$. Then $(S = f^{-1}(q), \iota)$ is a regular submanifold of M, ι = inclusion : $S \to M$,
  — $\dim S = \dim M - \dim N$. (for proof 'localize')
- <u>Propn 2</u>:- Let $f: M \to N$ be smooth an $q \in N$ a regular value, $(S = f^{-1}(q), \iota)$ the corresp. regular submanifold. Then, for $p \in S$, T_pS is a <u>subspace</u> of T_pM and

  $T_pS = \ker Df(p) \subseteq T_pM$.

- <u>Remarks on the proof of Prop. 2</u>: By Prop. 1, (S, ι) is a regular submanifold of M, hence $D\iota(p): T_pS \to T_pM$ is 1-1. Hence we may identify T_pS with $D\iota(p)(T_pS)$.
- f is constant on $S \Rightarrow T_pS \subseteq \ker Df(p)$ (details you should supply).
- $Df(p)$ is onto : $T_pM \to T_pN$ (f is a submersion at p)

$\therefore \dim \ker Df(p) = \dim M - \dim N = \dim T_pS$. ☒