Lecture-11

<u>Vector fields</u> <u>on manifolds</u> (contd): Recall that for M <u>smooth</u>, a <u>smooth</u> vector field on M is a map $X: M \to TM = \bigcup_p T_pM$, $X(p) \in T_pM$, $p \in M$ and for <u>any</u> coordinate chart U on M with coordinates $x_1, \dots, x_m$, writing $X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i}\Big|_q$ for any $q \in U$, the functions f_i are <u>smooth</u> with respect to any local chart in the <u>smooth</u> atlas of M. It is illustrative to discuss these in the C^r-category, since some properties of vector field are more visible:

• For a C^{k+1}-manifold M, $k \geq 1$, we talk of C^k-vector fields. For any chart (U, φ) on M, the map $p \mapsto \frac{\partial}{\partial x_i}\Big|_p$, $p \in U$, $1 \leq i \leq \dim M = m$, is a <u>C^k-vector field</u> on U

Where we may define this notion by any of the:

(i) $X: M \to TM$ is C^k.

(ii) For any chart (U, φ) and $f_1, \dots, f_m$ the unique functions on U defined by

$$X|_U = \sum_i f_i \frac{\partial}{\partial x_i}.$$ Then f_i is C^k $\forall i$.

(iii) $\forall$ open $U \subseteq M$ & $f \in C^k(U)$ we have $\to$

$X(f) \in C^{k-1}(U)$. Where, recall that, for $p \in U$, $X(f)(p) := X(p)(f)$. (*)

We discussed the derivational properties of smooth vector fields on smooth manifolds and proved that any derivation $\delta : C^\infty(M) \to C^\infty(M)$ comes from a unique smooth vector field X on M, i.e. $\tilde{X} = \delta$.

- Given $X, Y \in \mathcal{X}(M)$, we can compose $X \circ Y$ and the following reasons compel us to consider $X \circ Y - Y \circ X =: [X, Y]$.

__Exercise__ :- 1. Let A be a ring and $\delta_1, \delta_2 : A \to A$ be derivations (i.e. additive & satisfy the product rule). Then, in general, $\delta_1 \circ \delta_2$ is not a derivation.

2. $[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \mathrm{Der}(A)$.

Even when M is smooth & $X, Y \in \mathcal{X}(M)$, there is, in general, no vector field Z on M with $Z(f) = X(Y(f))$ $\forall f \in C^\infty(M)$.

__Example__ :- Let $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ be vector fields on $M = \mathbb{R}^3$.

 Then, for any $f \in C^\infty(\mathbb{R}^3)$,

$$X(Yf) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial z} + x\frac{\partial^2 f}{\partial x \partial z}.$$

$$= \left(\frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial z} + x\frac{\partial^2}{\partial x \partial z}\right)(f)$$

Thus $X \circ Y$ may involve (and $Y \circ X$ too) second order diff. operators, hence for $p \in U$, $(X \circ Y)(p)$ may fail to be in T_pM. To rectify this, we consider $[X, Y](f) := X(Y(f)) - Y(X(f))$.

Then
- If X, Y are C^k vector fields on C^{k+1}-mfld M, then $[X, Y]$ is C^{k-1} and this is determined (i.e. is!) by $Z(f) = X(Y(f)) - Y(X(f))$ $\forall f \in C^{k-1}(M)$.

In the above example, $Y(X(f))$

$$= \left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right)\left(\frac{\partial f}{\partial x}\right) = \left(\frac{\partial^2}{\partial y \partial x} + \frac{x\partial^2}{\partial z \partial x}\right)(f)$$

Hence $[X, Y](f) = \dfrac{\partial}{\partial z}(f)$; $[X, Y] = \dfrac{\partial}{\partial z} \in T(\mathbb{R}^3)$

Lemma:- Let M be smooth, $X, Y \in \mathcal{X}(M)$, $p \in M$, $f \in C^\infty(M)$. Then

$$[X,Y](p)(f) = X(p)(Y(f)) - Y(p)(X(f)).$$

Proof:- By def^n, $[X,Y](f) \in C^\infty(M)$ and is given by $[X,Y](f)(p) = [X,Y](p)(f)$ and $[X,Y](f) = X(Y(f)) - Y(X(f))$. Also, we defined (identifying $\tilde{X} \equiv X$), $X(f)$ by $X(f)(p) = X(p)(f)$. We thus have

$$\big(X(Y(f)) - Y(X(f))\big)(p) = X(p)(Y(f)) - Y(p)(X(f))$$

and, ($X \mapsto \tilde{X}$ being additive),

$$[X,Y] = X \circ Y - Y \circ X \quad \& \quad [X,Y](f)(p)$$
$$\shortparallel$$
$$= [X,Y](p)(f) = \big(X(Y(f)) - Y(X(f))\big)(p).$$

$\boxtimes$

Def^n:- Given $X \in \mathcal{X}(M)$, we denote

$$\mathcal{L}_X : C^\infty(M) \to C^\infty(M), \quad \mathcal{L}_X(f)(p) := X(p)(f)$$

and call $\mathcal{L}_X(f)$ the <u>Lie-derivative of f with respect to X</u>.

Remark:- Recall that for $f \in C^\infty(M)$

We defined $df : M \to \bigcup_p T_pM^*$, $df(p) \in T_pM^*$, for $v \in T_pM$, $df(p)(v) = v(f)$. We have, by the above def^n, $\mathcal{L}_X(f)(p) = X(p)(f)$.

Hence $\mathcal{L}_X(f)(p) = df(X(p))$, $p \in M$, $f \in C^\infty(M)$.

- $\mathcal{L}_{aX+bY} = a\mathcal{L}_X + b\mathcal{L}_Y$, $a, b \in \mathbb{R}$.
- When $M = \mathbb{R}^n$ and $X(p) = v$, $\mathcal{L}_X(f)(p)$ is the directional derivative of f at p in the direction v.

- $\mathfrak{X}(M)$ is a $C^\infty(M)$-module. Though $[X, Y]$ is $\mathbb{R}$-bilinear in both variables, it is not $C^\infty(M)$-bi-linear. We have

$$[fX, gY] = fg[X, Y] + (f \cdot X(g))Y - (g \cdot Y(f))X.$$

In particular $\mathcal{L}_{fX} Y \neq f\mathcal{L}_X Y$, where $\mathcal{L}_X(Y) := [X, Y]$ is defined to be the Lie derivative of Y with respect to X.

- For $X = \sum_i f_i \frac{\partial}{\partial x_i}$, $Y = \sum_j g_j \frac{\partial}{\partial x_j}$ on (U, φ),

$$[X, Y] = \sum_j \left(\sum_i \left[f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i} \right] \right) \frac{\partial}{\partial x_j}.$$

<u>Exercise</u>: 1. Let A be a (commutative) ring, $\delta_1, \delta_2 \in Der(A)$. Then $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in Der(A)$.

2. $[\delta_1, \delta_2] = -[\delta_2, \delta_1]$; for $\delta_3 \in Der(A)$,

$$[\delta_1, [\delta_2, \delta_3]] + [\delta_2, [\delta_3, \delta_1]] + [\delta_3, [\delta_1, \delta_2]] = 0.$$

The identity (2) is called the Jacobi identity.

- For $X, Y, Z \in \mathcal{X}(M)$ it thus follows $[X, Y] = -[Y, X]$ & Jacobi identity holds.

<u>Def^n</u>:- A <u>real</u> Lie algebra is an $\mathbb{R}$-vector space L, with an $\mathbb{R}$-bilinear map $[\,,\,]: L \times L \to L$ satisfying (i) $[x, y] = -[y, x]$ $\forall\, x, y \in L$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ $\forall\, x, y, z \in L$.

<u>Examples</u>. 1. Any vector space L over $\mathbb{R}$ with $[\,,\,] \equiv 0$ is a Lie algebra, called an abelian Lie algebra.

2. A be any associative $\mathbb{R}$-algebra (e.g. $A = M_n(\mathbb{R})$). Define $[a, b] = ab - ba$. Then $(A, [\,,\,])$ is a Lie algebra.

3. $sl(n, \mathbb{R})$, $O(n, \mathbb{R})$, $su(n)$, $u(n)$ are Lie algebras,

here we use the same Lie bracket as (2).

<u>Remark</u> :- Let M, N be smooth manifolds, $\varphi : M \to N$ be smooth, let $X \in \mathcal{X}(M)$. Define, for $p \in M$, $q = \varphi(p) \in N$,

$Y(q) = D\varphi(p)(X(p))$. One can ask: if φ is a homeomorphism, does this give a <u>smooth</u> vector field on N?.

<u>Example</u> :- Let $M = N = \mathbb{R}$ with x as the canonical coordinate function. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the map $\varphi(t) = t^3$. Then φ is smooth & a homeomorphism. Define

$$Y(u) = X(t) = D\varphi(t)\left(\frac{d}{dx}\Big|_t\right), \quad u = t^3$$

$$= \frac{d\varphi}{dx}\Big|_t \cdot \frac{d}{dx}\Big|_{\varphi(t)} = 3t^2 \frac{d}{dx}\Big|_u$$

So $Y = 3x^{2/3}\frac{d}{dx}$. The function $x \mapsto x^{2/3}$ is not diff at 0, hence Y is not smooth.

<u>Def</u>^{n} :- Let M, N be smooth and $\varphi : M \to N$ smooth. We say $X \in \mathcal{X}(M)$ & $Y \in \mathcal{X}(N)$ are <u>φ-related</u> if $D\varphi(p)(X(p)) = Y(\varphi(p))\ \forall p \in M$.

so $D\varphi \circ X = Y \circ \varphi$; i.e. $D\varphi(p)(X(p)) = Y(\varphi(p))\ \forall p$.

Exercise: If $\varphi: M \to N$ is a diffeomorphism, then for $X \in \mathfrak{X}(M)$, $p \in M$, $q = \varphi(p)$, the vector field $Y: q \mapsto D\varphi(p)(X(p))$ is smooth.

Lemma:- $\varphi: M \to N$ smooth, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ then X, Y are φ related $\iff$ $\forall\, g \in C^\infty(N)$, $X(g \circ \varphi) = Y(g) \circ \varphi$.

Pf:- X, Y are φ-related $\iff D\varphi \circ X = Y \circ \varphi$

We have, for $p \in M$ & $g \in C^\infty(N)$,

$$D\varphi(p)(X(p))(g) = Y(\varphi(p))(g)$$

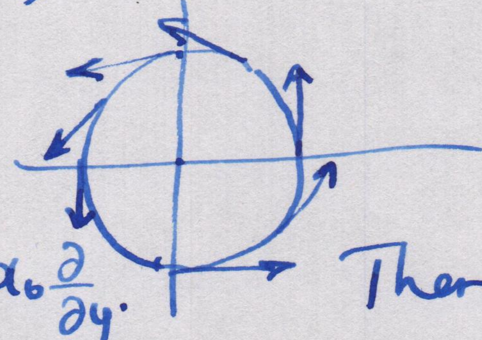
$$\iff X(p)(g \circ \varphi) = Y(g)(\varphi(p))$$

$$\iff X(g \circ \varphi)(p) = (Y(g) \circ \varphi)(p)$$

$$\iff X(g \circ \varphi) = Y(g) \circ \varphi. \qquad \boxtimes$$

Example from the book:- Let $\varphi: \mathbb{R} \to S^1 \subseteq \mathbb{R}^2$ be $\varphi(t) = (\cos t, \sin t)$. Let $X \in \mathfrak{X}(\mathbb{R})$ be $X = \frac{d}{dt}$, t coord on $\mathbb{R}$, $Y \in \mathfrak{X}(\mathbb{R}^2)$ be

$$Y = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$



i.e. $Y((x_0, y_0)) = -y_0 \frac{\partial}{\partial x} + x_0 \frac{\partial}{\partial y}$. Then $Y|_{S^1}$ is a vector field on S^1, φ related to X

Exercise: Write details.

Vector fields on Lie groups :- In the last example we had $M = S^1$, a Lie group and the vector field Y on it. Note that $Y(p)$ was obtained from $Y(e)$, $e = (1,0) =$ identity elem of S^1, by 'rotation' corresponding to $p \in S^1$. To be precise, recall that in a Lie group G the mult. $\mu : G \times G \to G$ & $i : G \to G$ resp. the mult. & inversion, are smooth. It follows that for $g \in G$, the maps $L_g : G \to G$, $x \mapsto gx$ and $R_G : x \mapsto xg$, are both smooth.

Let $v \in T_eG$ and $g \in G$. Define X by

$$X(g) = DL_g(v) \equiv DL_g(e)(v) \in T_g(G).$$

- Let (U, x) be a chart $\ni e$ and (V, y) a chart $\ni g$. Write $v = \sum a_j \frac{\partial}{\partial x_j}$, $a_j \in \mathbb{R}$.

Then $X(g) = DL_g^{(e)}(v) = DL_g^{(e)}\left(\sum a_j \frac{\partial}{\partial x_j}\right)$

$$= \sum_j a_j DL_g^{(e)} \frac{\partial}{\partial x_j} = \cancel{\sum_j a_j \sum_i DL_g(e)(y_i) \frac{\partial}{\partial y_i}}$$

$$= \sum_j a_j \sum_i DL_g(e)\left(\frac{\partial}{\partial x_j}\right)(y_i) \frac{\partial}{\partial y_i}$$

$= \sum_{j,i} a_j \frac{\partial}{\partial x_j}(y_i \circ L_g) \frac{\partial}{\partial y_i}$ on some open subset of V containing g.

Now L_g is a diffeomorphism, hence $y_i \circ L_g$ is C^∞ and, for each j, $a_j \frac{\partial}{\partial x_j}(y_i \circ L_g)$ is C^∞. Hence X is C^∞, i.e. $X \in \mathcal{X}(G)$.

→ Intuitively we feel that X, which we constructed by translating around v on G, should be 'translation invariant'.

• __Def__^{n}:- We call a vector field $X \in \mathcal{X}(G)$ __left invariant__ if X is L_g related to itself $\forall g \in G$, i.e. $DL_g \circ X = X \circ L_g$ $\forall g \in G$.

• If X is left invariant, then $DL_g(X(e)) = X \circ L_g(e) = X(g)$.

• If Y is another left inv. vector field $\in \mathcal{X}(G)$ with $Y(e) = X(e)$, then $Y = X$. Since

$$X(g) = DL_g(X(e)) = DL_g\, Y(e) = Y(g).$$

• $\mathcal{G}$ = Set of all left inv. vector fields on G. is an $\mathbb{R}$-vector subspace of $\mathcal{X}(G)$.

$\mathcal{G} \to T_e(G)$, $X \mapsto X(e)$ is an $\mathbb{R}$-isom.

Hence $\dim \mathcal{G} = \dim G = \dim T_e G$.

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