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Lecture-12

Summary of work so far :- Let us quickly summarize what we have discussed so far in the course.

- Differentiable manifolds : To study geometric objects like curves, surfaces intrinsically, not as embedded objects in some Euclidean space. Locally the discussion can be brought down to the Euclidean setting using the coordinate chart maps.
- Smooth functions : between smooth manifolds are defined using the notion in Euclidean spaces, using chart maps, transition functions facilitate this.
- Need of calculus on manifolds : Geometry is bothered by studying shapes of objects. One way to do that is to see how the object 'turns' or 'curves' around each of its points. From the Euclidean experience, this is done by measuring the 'rate of change of tangent spaces' on the object (smooth!), which naturally requires the notion of derivative of a smooth map; To talk of 'curvature', metric etc.
- Tangent vectors / Tangent spaces : We need 'direction' to talk of derivative of a function. Since we do not have an ambient Euclidean space, we achieve this by borrowing vectors from $\mathbb{R}^n$ locally:

- We used curves in the Euclidean case to define tangent vectors; if $U \subseteq \mathbb{R}^n$ open, $p \in U$, $\gamma : (-\epsilon, \epsilon) \to U$ a curve in U, $\gamma(0) = p$. Then $\dot{\gamma}(0) = (\dot{\gamma}_1(0), \dots, \dot{\gamma}_n(0))$ is defined as a tangent vector to U at p.

This inspires the def^n: $T_p(U)$ = set of equiv. classes of curves in U, under the equiv. rel^n $\sigma \sim \tau$ if $\sigma(0) = \tau(0) = p$, $\dot{\sigma}(0) = \dot{\tau}(0)$; $[\sigma] :=$ class of σ.

Tangent vectors as differential operators: For $U \subseteq \mathbb{R}^n$ as above, we proved $T_p(U) = \mathbb{R}^n$, since any $v \in \mathbb{R}^n$ (assuming p = origin) can be realized as $\dot{\gamma}(0)$, e.g. (γ a straight line) $\cap\, U$.

Given $v \in \mathbb{R}^n$ and $f \in C^\infty(p)$, we think of v as a diff. operator by setting $v(f) :=$ directional derivative of f at p, in the direction of v.

This gives a natural $T_p(U) \equiv \mathbb{R}^n \equiv \mathrm{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R})$.

- The directional derivative $D_v f(p)$ can be computed as $\frac{d}{dt}(f \circ \gamma(t))\Big|_{t=0}$ for any curve γ on U with $\gamma(0) = p$, $\dot{\gamma}(0) = v$. Hence the operator v on $C^\infty(p)$ can be identified with the equiv. class $[\gamma]$ as above; i.e.

$$T_p(U) \equiv \mathrm{Der}_{\mathbb{R}}(C^\infty(p), \mathbb{R}) \equiv \{[\gamma] \mid \gamma(0) = p\}$$

- The first identification is because we proved (Lecture 8) that any $\delta \in Der_{\mathbb{R}}(C^\infty(p), \mathbb{R})$, $\exists!\, v \in \mathbb{R}^n \equiv T_p(U)$, s.t. $\delta = [v]$.
- Under these identifications, the (standard) basis vectors of $\mathbb{R}^n$, e_i correspond to the directional derivative operator at $(p=0)$ in the direction e_i, i.e. $\frac{\partial}{\partial x_i}$. So $\left\{\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\right\}$ from a basis of $T_p(U)$.
- We <u>define</u> for M smooth, $p \in M$, a tangent vector v at p, as an element of ~~$C^\infty(p)$~~ $Der_{\mathbb{R}}(C^\infty(p), \mathbb{R})$, i.e. $T_pM := Der_{\mathbb{R}}(C^\infty(p), \mathbb{R})$.
- If $(U, \varphi) \ni p$ is a coordinate chart, $\varphi(p) = 0$ and $\{u_i\}_{1 \le i \le n}$ the canonical coordinates in $\mathbb{R}^n$, $x_i := u_i \circ \varphi$, We <u>define</u> for $f \in C^\infty(p)$,

$$\frac{\partial}{\partial x_i}\Big|_p (f) := \frac{\partial}{\partial u_i}(f \circ \varphi^{-1})(0) \equiv \frac{\partial}{\partial u_i}\Big|_0 (f \circ \varphi^{-1}).$$

$\left\{\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\right\}$ form a basis of $T_p(M)$; for $v \in Der_{\mathbb{R}}(C^\infty(p), \mathbb{R}) = T_p(M)$, we have

$$v = \sum_i v(x_i) \frac{\partial}{\partial x_i}\Big|_p$$

- For $\gamma : (-\epsilon, \epsilon) \to M$ smooth, $\gamma(0) = p$, define

12-4. $\dot{\gamma}(0) \in T_pM$ by $\dot{\gamma}(0)(f) := \frac{d}{dt}(f\circ\gamma)(0) = \frac{d}{dt}\Big|_{t=0}(f\circ\gamma)$,
here $f \in C^\infty(p)$ is arbitrary.

- <u>Derivatives</u> of <u>smooth maps</u>: M, N smooth, $f: M \to N$ smooth, $p \in M$. Then we would like to generalize the notion of derivative $f: \mathbb{R}^m \to \mathbb{R}^n$; which, for $p \in \mathbb{R}^m$, is a linear map: $T_p\mathbb{R}^m \to T_{f(p)}\mathbb{R}^n$.

We <u>define</u> $Df(p): T_pM = \mathrm{Der}_{\mathbb{R}}(C^\infty_M(p), \mathbb{R}) \to T_{f(p)}N = \mathrm{Der}_{\mathbb{R}}(C^\infty_N(f(p)), \mathbb{R})$

by ~~Af(f)~~ $Df(p)(v)(g) = v(g\circ f)$ for $g \in C^\infty_N(f(p))$ arbitrary. <u>Equivalently</u> if $v \in T_pM$, $\exists$ a smooth curve γ on M, $\gamma(0) = p$ & $v = \dot{\gamma}(0)$ (as an elem. of $\mathrm{Der}_{\mathbb{R}}(C^\infty_M(p), \mathbb{R})$).
Then $f\circ\gamma \subseteq N$ is smooth, $f\circ\gamma(0) = f(p)$.
Define $Df(p)(v) = (\dot{f\circ\gamma})(0)$. Then $Df(p): T_pM \to T_{f(p)}N$ is a linear map, gen. the Euclidean notion.

- <u>Jacobian matrix</u> :- $Df(p) \equiv \left(\frac{\partial}{\partial x_i}\Big|_p (y_j\circ f)\right)_{\substack{1\le i\le m\\ 1\le j\le n}}$

$(V, \psi) \ni q = f(p)$ a coord. chart in N with $\{y_1, \dots, y_n\}$ as coord. functions on V.

- $D(g\circ f)(p) = Dg(f(p))\circ Df(p)$ (Chain rule
$p \in M \xrightarrow{f} N \xrightarrow{g} W$.

12-5 • <u>Special case</u>: $M = (-\epsilon, \epsilon) \subseteq \mathbb{R}$, chart $u : t \mapsto t$.
$T_0 M = \mathbb{R} \cdot \frac{d}{du}\big|_0$. If N smooth, $f \in C^\infty(N)$, $p \in N$,
$v \in T_p N$. Then $Df(p)(v) \in T_{f(p)}\mathbb{R} = \mathbb{R} \cdot \frac{d}{du}\big|_{f(p)}$ &

$$Df(p)(v) = v(f) \cdot \frac{d}{du}\Big|_{f(p)}.$$

• <u>Cotangents, Cotangent space</u>: M smooth, $p \in M$.
$(T_p M)^* =$ Cotangent space to M at p. For
$f \in C^\infty(M)$, $Df(p) : T_p M \to T_{f(p)}\mathbb{R} = \mathbb{R}$, so
$Df(p) \in (T_p M)^*$ and for $v \in T_p M$,
$Df(p)(v) = v(f)$. When treated as an element of $(T_p M)^*$, we denote $Df(p)$ by
$df(p)$. We write $dx_i(p)$ for x_i coords around p, then $dx_i(p) = \left(\frac{\partial}{\partial x_i}\big|_p\right)^* \in T_p M^*$
make the dual basis.

• $df : M \to \bigcup_p T_p M^*$, $f \in C^\infty(M)$ has a <u>local expression</u>

$$df(x) = \frac{\partial f(x)}{\partial x_1} dx_1 + \dots + \frac{\partial f(x)}{\partial x_m} \cdot dx_m$$

for $x \in U$, $(U, \varphi) \ni p$ a local chart.

• Defined <u>Submanifolds</u>, <u>submersions</u>, <u>immersions</u>.
Submersions & immersions on manifolds are canonical locally.

- <u>Vector fields</u>: Tangent vectors are 'local' diff. operators, acting on the algebra of (germs of) locally smooth functions on M. Vector fields are global diff operators, acting on the alg. of globally smooth functions on M, i.e. elements of $\mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$. These are maps $X: M \to T(M) = \bigcup_p T_pM$, $X(p) \in T_pM$ such that, for any coord. chart $U \ni p$ with coordinates $x_1, .., x_m$, we have

$$X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i}\Big|_q \quad , q \in U;$$

f_i are smooth.

- $\mathcal{X}(M)$ = $\mathbb{R}$-vector space of smooth vect. fields on M, is closed under the $\mathbb{R}$-bilinear map $[\ ,\ ]: \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$, $[X, Y]$ is given by $[X, Y] = X \circ Y - Y \circ X$; for $p \in M$, $f \in C^\infty(M)$, this yields

$$[X, Y](f)(p) = [X, Y](p)(f).$$
$$= X(p)(Y(f)) - Y(p)(X(f)).$$

- <u>Lie derivatives</u>: For $X \in \mathcal{X}(M)$, when treated as an $\mathbb{R}$-derivation: $C^\infty(M) \to C^\infty(M)$, We write $\mathcal{L}_X(f) := X(f)$, $f \in C^\infty(M)$ and call it the Lie derivative of f with respect to X. When $p \in M$, we have $X(p) \in T_pM$. Given $f \in C^\infty(M)$, $p \in M$, what is $\mathcal{L}_X(f)(p)$? Following notations, we see that

$$\mathcal{L}_X(f)(p) = X(p)(f) \leftarrow \text{directional}$$

derivative of f at p, in the direction $X(p)$.

Hence we may think of a vector field X as built from the local directional derivative operators $X(p)$, $p \in M$.

• Lie algebras and vector fields :- A Lie alg/$\mathbb{R}$ is a vector space L over $\mathbb{R}$, with a $\mathbb{R}$-bilinear map $[\,,\,]$, called the Lie bracket, $[\,,\,] : L \times L \to L$ satisfying $[x,y] = -[y,x]$; Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \quad x,y,z \in L.$$

• $(\mathfrak{X}(M), [\,,\,])$ is a Lie algebra.

Exercise :- 1. L be a Lie alg, $[\,,\,]$ its Lie bracket. Let $x \in L$ be fixed. Define $ad(x) : L \to L$ by $ad(x)(y) := [x,y]$. Prove that $ad(x)$ is a derivation on L, i.e. satisfies the

12-8 Product rule $ad(x)([y,z]) = [ad(x)(y), z] + [y, ad(x)(z)]$.

$\to$ Jacobi identity is equivalent to the above.

2. L be a Lie algebra. We then have the $\mathbb{R}$-linear map $ad(x): L \to L$ for any $x \in L$. Thus an $\mathbb{R}$-linear map $ad: L \to End_{\mathbb{R}}(L)$ $x \mapsto ad(x)$. Prove that 'ad' is a Lie alg. homomorphism, i.e. is additive, ~~$\mathbb{R}$-linear~~ &

$$ad([x,y]) = \underbrace{ad(x) \circ ad(y) - ad(y) \circ ad(x)}_{= [ad(x), ad(y)] \text{ in } End_{\mathbb{R}}(L)}$$

$\forall\ x, y \in L$.

• <u>Vector fields on Lie groups</u>:- On a Lie group G, the translation maps $L_g: G \to G$ are diffeomorphisms. Given $v \in T_e G$, $g \in G$ we have $DL_g(e): T_e G \to T_g G$, so $DL_g(e)(v) \in T_g G$. We define a vector field X on G by $X(g) := DL_g(e)(v)$.

Then X is <u>left-invariant</u>; ~~see~~.

Call $Y \in \mathfrak{X}(G)$ left-invariant if

$$DL_g \circ Y = Y \circ L_g.$$

$\rightsquigarrow$

→ Any left invariant vector field on G is completely determined by its value at e.

→ $\mathcal{G} :=$ $\mathbb{R}$-vector space of all left-inv. smooth vector fields on G.

The map $\mathcal{G} \to T_e G$, $X \mapsto X(e)$ is an isomorphism of vector spaces; hence $\dim \mathcal{G} = \dim T_e G = \dim G$. ☒

<u>Remark</u>: What we did with left transl^n maps, can be done, similarly, with right transl^{n}s on G.

<u>Recall</u>: Given $\varphi: M \to N$ smooth, $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$, we say Y is φ-related to X if $D\varphi \circ X = Y \circ \varphi$, i.e.

$$D\varphi(p)(X(p)) = Y(\varphi(p)) \quad \forall\, p \in M.$$

We need

<u>Lemma</u>:- $X_1, X_2 \in \mathcal{X}(M)$ be φ-related to $Y_1, Y_2 \in \mathcal{X}(N)$ respectively. Then $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.

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Pf: We proved $X \in \mathfrak{X}(M)$ is φ-rel to $Y \in \mathfrak{X}(N)$ $\iff X(g\circ\varphi) = Y(g)\circ\varphi \ \forall g \in C^\infty(N)$. We compute

$$X_1X_2(g\circ\varphi) = X_1(X_2(g\circ\varphi)) = X_1(Y_2(g)\circ\varphi)$$
$$= Y_1(Y_2(g))\circ\varphi$$
$$= Y_1Y_2(g)\circ\varphi$$

and similarly $X_2X_1(g\circ\varphi) = Y_2Y_1(g\circ\varphi)$.

Thus $[X_1, X_2](g\circ\varphi) = [Y_1, Y_2](g\circ\varphi), \forall g$.

$\therefore$ $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$. ⊠

Lie algebra of a Lie group: We use the lemma to $\mathfrak{g} = \{$left inv. vector fields on $G\}$.

$$= \{X \in \mathfrak{X}(G) \mid X \text{ is } L_g\text{-related to itself } \forall g \in G\}.$$

Hence it follows that $\forall\ X, Y \in \mathfrak{g}$, $[X, Y] \in \mathfrak{g}$, i.e. $\mathfrak{g}$ is a Lie-Subalgebra of $\mathfrak{X}(G)$, called the Lie algebra of G, also denoted by $\mathrm{Lie}(G)$. This plays a big role in the structure theory of Lie groups & their classification.

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