Lecture-13

<u>Lie algebra of a Lie group</u>:- We ended Lecture-11 by proving: if G is a Lie group and $\mathfrak{g}$ its Lie algebra, i.e. $\mathfrak{g} = \{X \in \mathfrak{X}(G) \mid X \text{ is left inv.}\}$, then the map $\mathfrak{g} \to T_eG$, $X \mapsto X(e)$ is an isomorphism of vector spaces.

→ If $X, Y \in \mathfrak{g}$, then $[X,Y] \in \mathfrak{g}$ and $(\mathfrak{g}, [\,,\,])$ is a Lie algebra. We transfer this Lie algebra structure to T_eG by defining, for $X, Y \in \mathfrak{g}$; $[X(e), Y(e)] := [X,Y](e)$.

→ This gives T_eG a Lie algebra structure and we sometimes identify $\mathfrak{g}$ with T_eG via the above; $\mathfrak{g} \to T_eG$, $X \mapsto X(e)$ is a Lie algebra isomorphism.

<u>Example</u>:- Consider the Lie group $G = GL_n(\mathbb{R})$. Since $G \subseteq \mathbb{R}^{n^2} \simeq M_n(\mathbb{R})$ is an open set, namely "$\{X \mid \det X \neq 0\}$, we have $\dim G = n^2$ and T_IG can be identified with $M_n(\mathbb{R})$.

The (global) coordinates on $\mathbb{R}^{n^2}$ are to be labelled as $x_{11}, x_{12}, \dots, x_{n1}, \dots, x_{nn}$. ⇝

Then, on GL_n, $\left\{ \frac{\partial}{\partial x_{11}}\Big|_I, \ldots, \frac{\partial}{\partial x_{ij}}, \ldots, \frac{\partial}{\partial x_{nn}}\Big|_I \right\}$ from a basis of $T_I(GL_n)$. We know that any $v \in T_I G$ has ! expression $v = \sum_{ij} v_{ij} \frac{\partial}{\partial x_{ij}}\Big|_I$ and $v_{ij} = v(x_{ij})$. By the identification of $T_I G$ with $\mathbb{R}^{n^2}$ & in turn with $M_n(\mathbb{R})$, we get

$$\underset{\mathrm{Lie}(GL_n)}{T_I GL_n} \longrightarrow M_n(\mathbb{R}) .$$
$$v \longmapsto (v(x_{ij}))_{ij}$$

We have an issue here to settle! We have a Lie algebra structure on $T_I GL_n$, a Lie alg. structure on $M_n(\mathbb{R})$. The above is, as of now, only a vector space map. We'll prove: <u>the above is a Lie alg. isom.</u>

<u>Proof</u>:- Let $X, Y \in \mathrm{Lie}(GL_n)$; so these are left-inv. vector fields on GL_n. These give $X(e), Y(e) \in M_n(\mathbb{R})$ by the above recipe, also we have $[X, Y](e) \in M_n(\mathbb{R})$, where $[X, Y]$ is the Lie bracket on $\mathfrak{X}(GL_n)$. To show the above map is a Lie alg map, it suffices to check: $[X, Y](e)_{ij} = [X(e), Y(e)]_{ij}$.

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Now $[X,Y](e)_{ij} = [X,Y](e)(x_{ij})$
$$= X(e)Y(x_{ij}) - Y(e)X(x_{ij}).$$

We have $X = \sum X(x_{ij}) \frac{\partial}{\partial x_{ij}}$, $Y = \sum Y(x_{ij}) \frac{\partial}{\partial x_{ij}}$.

Also, for $a \in GL_n$,
$$X(x_{ij})(a) = X(e)(x_{ij} \circ L_a) \text{ by left inv.}$$

$$\rightarrow \begin{cases} X(x_{ij})(a) = DL_a(X(e))(x_{ij}) = X(e)(x_{ij} \circ L_a), \\ Y(x_{ij})(a) = Y(e)(x_{ij} \circ L_a). \end{cases}$$ We need to

Compute also $x_{ij} \circ L_a(g)$, for $g \in GL_n$
$$\mathrel{/\!/}$$
$$x_{ij}(ag) = \sum_k x_{ik}(a) x_{kj}(g).$$ Therefore

$\rightarrow x_{ij} \circ L_a = \sum_k x_{ik}(a) x_{kj}$. Hence
$$Y(x_{ij})(a) = Y(e)(x_{ij} \circ L_a) = Y(e)\Big(\sum_k x_{ik}(a) x_{kj}\Big)$$
$$= \sum_k x_{ik}(a) Y(e)(x_{kj}).$$

Hence $Y(x_{ij}) = \sum_k x_{ik} \cdot Y(e)(x_{kj})$ as functions.

Finally, $[X,Y](e)_{ij} = X(e)\Big(\sum_k x_{ik} Y(e)(x_{kj})\Big)$
$$- Y(e)\Big(\sum_k x_{ik} X(e)(x_{kj})\Big) = \sum_k X(e)(x_{ik}) Y(e)(x_{kj})$$
$$- \sum_k Y(e)(x_{ik}) X(e)(x_{kj}) = [X(e), Y(e)]_{ij}. \quad \boxtimes$$

<u>Exercise</u>: Check that $\mathfrak{sl}_n, \mathfrak{u}(n), \mathfrak{su}(n), \mathfrak{o}(n), \mathfrak{so}(n)$ are all Lie subalgebras of the corresponding matrix algebra, i.e. each of the above is Closed under the matrix Lie bracket. Recall the following: Lie group / Lie algebra / Condition

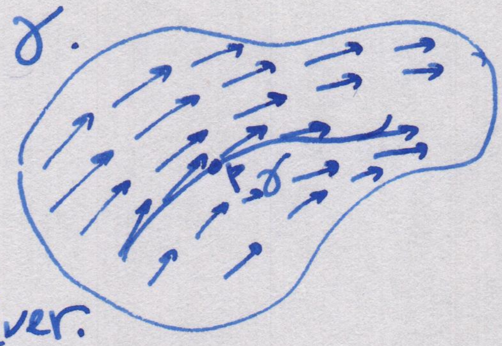
Lie group	Lie algebra	Condition
$GL_n(\mathbb{R})$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$	—
$SL_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{R})$	$\mathrm{Trace}(X) = 0$
$O(n, \mathbb{R})$	$\mathfrak{o}(n)$	$X^t = -X$
$SO(n, \mathbb{R})$	$\mathfrak{so}(n)$	" — "
$U(n)$	$\mathfrak{u}(n)$	$X^* = -X$
$SU(n)$	$\mathfrak{su}(n)$	$X^* = -X$ & $tr(X) = 0$
$SP(2n, \mathbb{R})$	$\mathfrak{sp}(2n)$	$JX^tJ = X$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

<u>Integral curves and flows of vector fields</u>:

Let M be a smooth manifold, $p \in M$. Let $v \in T_p(M)$. Then using local charts, we have shown that $\exists$ a smooth curve γ passing through p and $\dot{\gamma}(0) = v$. More precisely, $\gamma: (-\epsilon, \epsilon) \to M$, smooth, $\gamma(0) = p$ & $\dot{\gamma}(0) = v$ (as a derivation on $C^\infty(p)$). Let X be a smooth vector field on M. Then, the above gives us γ s.t. $\gamma(0) = p$ & $\dot{\gamma}(0) = X(p)$.

13-5. A priori, γ works for $p=\gamma(0)$, but if $Q\in M$ is on γ, say $\gamma(t_1)=Q$, we may not have $\dot\gamma(t_1)=X(Q)$.

$\boxed{\text{Def}^n}$ For $X\in\mathfrak{X}(M)$ and $p\in M$, a smooth curve γ is called an <u>integral curve of X, through p</u> if $\gamma(0)=p$ and $\dot\gamma(t)=X(\gamma(t))$ for all t in the domain of γ.



→ You may think of γ as the path traced by a blade of grass on the surface of a flowing river.

Thus, the curve γ as ~~ith~~ the def^n fits the vector field for an interval of time.

Integral curves are instrumental in studying vector fields on M, comparing tangent spaces at different points, in differential calculus on M.

Given $X\in\mathfrak{X}(M)$ & $p\in M$, does X admit an integral curve through p?

Suppose $(U,\varphi)\ni p$ is a coordinate chart on M, γ an integral curve of X through p. Then by def^n,
$$X(\gamma(t))=\dot\gamma(t)=\sum_{i=1}^{m}X(x_i)\frac{\partial}{\partial x_i}\Big|_{\gamma(t)}.$$
and
$$\dot\gamma(t_0)=\sum_{i=1}^{m}\frac{d}{dt}(x_i\circ\gamma)\Big|_{t=t_0}\cdot\frac{\partial}{\partial x_i}\Big|_{\gamma(t_0)}$$
→

Hence the question whether X admits an integral curve through p reduces to solving the system of ODE

$$\frac{d}{dt}(x_i \circ \gamma) = X(x_i), \; t \in (-\epsilon, \epsilon); \quad 1 \leq i \leq m$$

Using <u>local coordinates</u> around $p \in (U, \varphi)$, this reduces to finding $x : (-\epsilon, \epsilon) \to V = \varphi(U)$ satisfying $\frac{d}{dt} x = X(x)$, with initial value $x(0) = 0$ (*) (we assume $\varphi(p) = 0$);

We now apply:

<u>Thm (Prop. 1.9.1, Kum)</u> (Existense of sol^n's of ODE):
Let $V \subseteq \mathbb{R}^m$ be open, $X : V \to \mathbb{R}^m$ be Lipschitz; $\|X(x) - X(y)\| \leq M . \|x - y\| \; \forall x, y \in V$. Let $x_0 \in V$, $B(x_0, r) \subset U$, $C > 0$ be such that $\|X(x)\| \leq C \; \forall x \in B(x_0, r)$. Let $\epsilon < \min \left\{ \frac{1}{M}, \frac{r}{C} \right\}$. Then $\exists!$ C^1 curve $x : [-\epsilon, \epsilon] \to B(x_0, r)$ satisfying $\dot{x}(t) = X(x(t))$, $x(0) = x_0$. $\boxtimes$

to get a unique (local) solution to (*).

<u>Example</u>: Let $M = \mathbb{R}^2$, $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $p = (a, b)$. If we write $\gamma(t) = (x(t), y(t))$. Then we must solve $\dot{x} = -y$, $\dot{y} = x$, $x(0) = a$, $y(0) = b$

To find the integral of X through p, we solve $\frac{dx}{dt} = -y$, $\frac{dy}{dt} = x$, with initial values $x(0) = a$ & $y(0) = b$ resp. This has sol^n $x(t) = a\cos t - b\sin t$, $y(t) = a\sin t + b\cos t$. When $(a,b) \neq (0,0)$, we get a circle of radius $\sqrt{a^2+b^2}$. (do the details). (See Kum. for more).

<u>Flow of a vector field</u> :- We can rephrase the theorem on existence & uniqueness for sol^n of a system of ODE's to get better results:

<u>Thm</u> (Local flows): Let $V' \subseteq \mathbb{R}^m$ be open, $p \in V'$. Let $X : V' \to \mathbb{R}^m$ be C^k, $k \geq 1$. Then $\exists\, \epsilon > 0$ & open subset $V \subseteq V'$, $p \in V$, and a unique C^k-map $F : (-\epsilon, \epsilon) \times V \to V'$ such that
(i) $\frac{d}{dt}(F(t,x)) = X(F(t,x))$ (ii) $F(0,x) = x$ $\forall\, t \in (-\epsilon, \epsilon)$ & $x \in V$. (see Th. 2.8.9. Kum). ⊠

This gives, for a smooth manifold

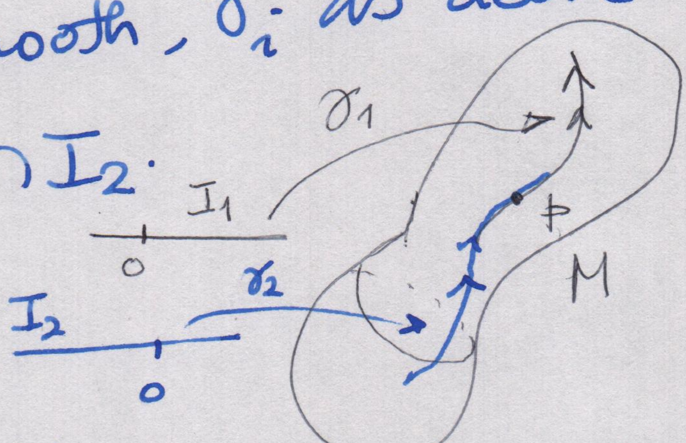
<u>Thm</u> (Existence of a local flow) :- Let X be a C^{k-1} vector field on a C^k-manifold M $(k \geq 2)$ and $p \in M$. There exists an open interval $I \ni 0$, open set $p \in U \subseteq M$ such that $\exists!$ local flow $F : I \times U \to M$ for X at p, F-C^{k-1},

i.e. if $F_1 : I\times U \to M$ & $F_2 : I\times U \to M$ are both local flows with domain $I\times U$, at p, then $F_1 = F_2$. $\boxtimes$

F is called the <u>flow</u> of the vector field X.
Hence a local flow for X is a family of integral curves for <u>all</u> points in some open set U around p, such that all these curves have the <u>same domain</u> I, independent of the initial condition $p\in U$.

We would like to compare two integral curves of X, $\gamma_1 : I_1 \to M$, $\gamma_2 : I_2 \to M$, $\gamma_i(0)=p$, $i=1,2$, $\dot\gamma_i(0) = X(p)$. We have

<u>Lemma</u>:- X, M be smooth, γ_i as above.
Then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.



<u>Proof</u>:- Consider $I_1\cap I_2$.
Since both are intervals, $p\in I_1\cap I_2 \Rightarrow I_1\cap I_2$ is an interval. The set $\{t\in I_1\cap I_2 \mid \gamma_1(t)=\gamma_2(t)\}$ is both open and closed in $I_1\cap I_2$. etc. $\boxtimes$

Let $\{\gamma_j\}_{j\in J}$ be the family of integral curves of X with $\gamma_j : I_j \to M$, $\gamma_j(0) = p$ $\forall j$. Then

The Lemma allows us a unique $\gamma : \bigcup_{j\in J} I_j \to M$ extending γ_j on I_j $\forall j$. Then γ is a smooth curve through p, is an integral curve of X, with largest (maximal) domain $I_p := \bigcup_j I_j$, we call γ the <u>maximal integral curve</u> of X through p, denote it by γ_p.

<u>Example</u>:- For $M = \mathbb{R}^2$ and $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $p = (x_0, y_0)$, γ_p is the circle

$\gamma_p(t) = (x(t), y(t))$, with

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0\cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \end{pmatrix}$$

and $I_p = \mathbb{R}$, for every $p \in \mathbb{R}^2$.

<u>Example</u>:- For $M = \mathbb{R}$, X be the vector field

$$X(x) = (1+x^2)\frac{\partial}{\partial x}.$$

<u>Exercise</u>:- Check that the maximal integral curve with initial condition $p=0$ is the curve $\gamma : (-\pi/2, \pi/2) \to \mathbb{R}$ given by

$$\gamma(t) = \tan t$$

We can ask the following natural question:
Given $p_0 \in M$, $\gamma_{p_0} : I_{p_0} \to M$ be the maximal integral curve for X with initial condition p_0. For $s \in I_{p_0}$ we have $p_1 = \gamma_{p_0}(s) \in M$.
Hence these is the maximal integral curve $\gamma_{p_1} : I_{p_1} \to M$ with initial condition p_1.
How are γ_{p_0} & γ_{p_1} related? We have

<u>Lemma</u>:- We have $I_{p_1} = I_{p_0} - s$ and
$$\gamma_{p_1}(t) = \gamma_{p_0}(t+s) \quad \forall\, t \in I_{p_0} - s.$$

In other words, the 'traces' $\gamma_{p_0}(I_{p_0})$ and $\gamma_{p_1}(I_{p_1})$ in M of the max integral curves γ_{p_0} and γ_{p_1} are the same; they differ only by a simple reparameterization
$$u = t + s.$$

<u>Pf</u>: Let $\sigma(u) = \gamma_{p_0}(u-s)$. Then $\dot{\sigma}(0) = \dot{\gamma}_{p_0}(-s)$
$= X(\gamma_{p_0}(-s))$. Hence σ is an integral curve of X through $\sigma(0) = \gamma_{p_0}(-s)$ etc. ⊠

<u>Def^n</u>: We call X complete if all max integral curves of X have domain $\mathbb{R}$, incomplete otherwise.