Lecture-14

Integral curves contd: One clarification on using the ODE theorem: if $f: V \to \mathbb{R}^m$ is continuously differentiable, $V \subseteq \mathbb{R}^d$ open. Then for every $p \in V$, $\exists\, r > 0$, such that all $\frac{\partial f_i}{\partial x_j}$ are bounded by some $M' > 0$. Then for $M := \sqrt{dm} \cdot M'$, one has, $\forall\, x, y \in B(0, r)$, $\|f(x) - f(y)\| \leq M \cdot \|x - y\|$. $\boxtimes$

We defined the notion of the maximal integral curve for a vector field and the notion of a complete vector field, i.e. that whose all max integral curves have domain $\mathbb{R}$, incomplete o/w.

Looking ahead; if X is a complete vector field on M, $\{\gamma_p\}_p$ its max integral curves, we'll see that there is a diffeomorphic action of $\mathbb{R}$ on M, i.e. a group homom. $\Phi_X : \mathbb{R} \to \mathrm{Diff}(M)$, induced from $\mathbb{R} \times M \to M$ given by $(t, p) \longmapsto \gamma_p(t)$. In particular if we write $\varphi_t(p) := \gamma_p(t)$, then

$$\varphi_s \circ \varphi_t = \varphi_{s+t}.$$

This is the analogue of translations on M, and we'll use this to compare tangent spaces at different points on M. $\boxtimes$

<u>Local flows</u> :- Let M be smooth and X a smooth vector field on M, $p \in M$. Then $\exists$ $\epsilon > 0$, open $U \subseteq M$, $p \in U$ and a <u>unique</u> smooth map $F : (-\epsilon, \epsilon) \times U \to M$ with

$$F(0, x) = x \ \forall x \in U, \ \frac{d}{dt}(F(x,t)) = X(F(x,t))$$

$\forall \ t \in (-\epsilon, \epsilon), x \in U$:

In other words $\exists$ a unique family of integral curves of X on U, <u>having the same domain</u> $(-\epsilon, \epsilon)$, passing through pts of U. $\boxtimes$

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With the above notation, denote for $(t, z) \in (-\epsilon, \epsilon) \times U$, $\varphi_t(z) := F(t, z)$. If γ_p be as befroe, $\gamma_p(t) = F(t, p)$ for $t \in I$.

$\to$ When X is complete, $\varphi_t : M \to M$ is a diffeomorphism, since

$$\varphi_s \circ \varphi_t(p) = \varphi_{s+t}(p), \quad \varphi_0(p) = p \ \forall p \in M.$$

<u>Proof</u> :- For a fixed s, we consider the curves $t \mapsto \varphi_s \circ \varphi_t(p)$ and $t \mapsto \varphi_{s+t}(p)$.

Then both have initial condition $= \varphi_s(p)$ and are integral curves of X. Hence they must be equal (having domains $= \mathbb{R}$).

Inverse of φ_t is φ_{-t} clearly. $\boxtimes$

→ <u>Def</u>^{n}:- $\{\varphi_t\}_{t\in\mathbb{R}}$ is called the one parameter group of diffeomorphisms associated to X.

Even when X is not complete one can define a local notion:

<u>Def</u>^{n}: A local 1-parameter family of <u>local</u> diffeomorphisms is a collection $\{U_\alpha, \varphi_t^\alpha, \epsilon_\alpha\}_\alpha$ satisfying: (i) U_α are open, $\bigcup_\alpha U_\alpha = M$

(ii) $\epsilon_\alpha > 0$ $\forall \alpha$ (iii) domain of $\varphi_t^\alpha \supseteq U_\alpha$ and the map $(t, z) \mapsto \varphi_t^\alpha(z)$ for $z \in U_\alpha$ & $t \in (-\epsilon_\alpha, \epsilon_\alpha)$, is smooth.

(iv) $\varphi_t^\alpha(z) = \varphi_t^\beta(z)$ $\forall z$ in $\text{domain}(\varphi_t^\alpha) \cap \text{dom}(\varphi_t^\beta)$ & t with $|t| < \min\{\epsilon_\alpha, \epsilon_\beta\}$.

(v) for t, s with $t, s, t+s \in (-\epsilon_\alpha, \epsilon_\alpha)$, $\varphi_t^\alpha \circ \varphi_s^\alpha$ is defined on an open set $\supseteq U_\alpha$ & $\varphi_t^\alpha \circ \varphi_s^\alpha = \varphi_{t+s}^\alpha$ on U_α. →

• <u>Examples / Exercise</u>: 1. Let $M = \mathbb{R}$. Then $\varphi_t : \mathbb{R} \to \mathbb{R}$ given by (i) $\varphi_t(x) = x + t$ (ii) $\varphi_t(x) = e^t x$ are both one-parameter groups of diffeoms.

2. $M = \mathbb{R}^2$, $\varphi_t((x,y)) = (\cos t x - \sin t y, \sin t x + \cos t y)$ is a one-parm. group of diffeoms on $\mathbb{R}^2$.

3. $M = \mathbb{R}^n$ and $A \in M_n(\mathbb{R})$ be fixed. Then $t \mapsto \exp(tA) =: \varphi_t$; $x \mapsto \varphi_t(x)$ is a one-param family of diffeoms.

• <u>Lemma</u>: Given a local 1-parameter family $\{\varphi_t^\alpha\}$ of M, $\exists$ a vector field $X \in \mathcal{X}(M)$ such that it generates the family.

<u>Pf</u>:- We define, for $p \in M$ & $X \in \mathcal{X}(M)$, $X(p)(f) := \frac{d}{dt}\Big|_{t=0} f \circ \varphi_t^\alpha$. Then X indeed is smooth. (?). <u>Let $p \in U_\alpha$</u> and define $\gamma_p : (-\epsilon_\alpha, \epsilon_\alpha) \to M$ by $\gamma_p(t) = \varphi_t^\alpha(p)$. Then γ_p is an integral curve to X, $\gamma_p(0) = p$ etc. $\boxtimes$

→ • For a given X, a local family of diffeoms is not unique nec.

14-5 <u>Example/exercise</u>:- Let $X \in \mathcal{X}(M)$, $\psi : M \to M$ a diffeomorphism. Let Y be the "push forward" vector field, i.e. for $q \in M$, $q = \psi(p)$, $Y(q) = D\psi(p)(X(p))$. Then $Y \in \mathcal{X}(M)$. If $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}_\alpha$ is a local one parameter family of diffeoms for X, then $\{(\psi(U_\alpha), \psi \circ \varphi_t^\alpha \circ \psi^{-1}, \epsilon_\alpha)\}_\alpha$ is one for Y.

<u>Complete vector fields</u>: We have some situations which yield complete fields.

- Let $X \in \mathcal{X}(M)$. ~~Lett~~ Assume that X admits a local one-param. family of local diffeoms $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}_\alpha$ with $\inf \epsilon_\alpha > 0$. Then X is complete.

<u>Proof</u> (sketch): Let $\epsilon = \inf \epsilon_\alpha > 0$ and define $\varphi_t(p) := \varphi_t^\alpha(p)$, if $p \in U_\alpha$, $t \in (-\epsilon, \epsilon)$. Extend φ_t to the whole of $\mathbb{R}$ by the recipe:

$$\varphi_t := \underbrace{\varphi_{t/n} \circ \cdots \circ \varphi_{t/n}}_{n\text{-fold}}, \quad n \text{ is s.t. } t/n \in (-\epsilon, \epsilon).$$

$\rightsquigarrow$

 Then φ_t is well defined and $\gamma_p(t) := \varphi_t(p)$ is a max integral curve for X with $\gamma_p(0) = p$ and domain $= \mathbb{R}$. Hence X is complete. $\boxtimes$

Vector fields on M compact / Lie groups:

Let M be compact & $X \in \mathcal{X}(M)$.

Lemma: X is complete.

Pf:- Let $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}$ be a local one-param. family for X. Compactness $\Rightarrow$ finitely many U_α's cover X, say $U_{\alpha_1}, \ldots, U_{\alpha_r}$. Then $\{(U_{\alpha_i}, \varphi_t^{\alpha_i}, \epsilon_{\alpha_i})\}$ is a 1-param family for X with $\inf \epsilon_{\alpha_i} > 0$. Hence X is complete. $\boxtimes$

→ If M be smooth & $X \in \mathcal{X}(M)$ have compact support, then X is complete. This follows by working with an open subset containing the support of X, instead of M.

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Lemma :- Let G be a Lie group and $X \in \mathfrak{g}$ i.e. X is a left-inv. vector field on G. Then X is complete.

Proof :- Let $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}_\alpha$ be a 1-param. family for X as above. Let $e \in G$ be the identity elet of G, $e \in U_{\alpha_0}$. Then the family $\{(L_g(U_{\alpha_0}), L_g \circ \varphi_t^{\alpha_0} \circ L_{g^{-1}}, \epsilon_{\alpha_0}), g \in G\}$ is a local 1-param. family for $DL_g(X) = X$, as X is left-inv. The criterion for completeness is thus satisfied by this family; hence X is complete. $\boxtimes$

Exponential map :- Let M be smooth and X a complete smooth vector field on M. Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be the 1-param group of diffeoms associated to X. We define the exponential map of X by $\mathrm{Exp}_X(p) := \varphi_1(p) = \gamma_p(1)$,

so $\mathrm{Exp}_X : M \longrightarrow M$. $\rightsquigarrow$

Examples :- 1. Let $A = (a_{ij}) \in M_n(\mathbb{R})$. Consider the vector field $\hat{A}$ on $\mathbb{R}^n$ defined by

$$\hat{A} = \sum_{i,j} a_{ij}\, x_j \frac{\partial}{\partial x_i}.$$

Consider $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by $F(t,x) = e^{tA}(x)$, where recall $e^{tA} = \sum_j \frac{t^j}{j!} A^j$. Clearly F is smooth.

This defines a flow for $\hat{A}$. Setting

$$\varphi_t(x) = e^{tA}x \quad (x \text{ a column vector}),$$

we have $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$, $\varphi_0 = 1$.

Let $\gamma_p(t) = F(t,p) = e^{tA}p$. Then for $f \in C^\infty(\mathbb{R}^n)$,

$$\dot{\gamma}_p(0)(f) = \frac{d}{dt}\Big|_{t=0} f(e^{tA}p) = \sum_j \frac{\partial f}{\partial x_j}(Ap)_j$$

$$= \sum_{j,k} a_{jk} p_k \frac{\partial f}{\partial x_j}, \text{ where}$$

$p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$. Hence $\dot{\gamma}_p(0) = \hat{A}(p)$.

Thus $\operatorname{Exp}_{\hat{A}}(p) = \varphi_1(p) = \gamma_p(1) = e^{A} \cdot p$ $\boxtimes$

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2. Lie groups: G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$.

We saw that any $X \in \mathfrak{g}$ is complete, so Exp_X makes sense. We define (note the confusing notation!)

$$\exp(X) := \mathrm{Exp}_X(e) = \varphi_1(e),$$

and more generally,

$$\exp(tX) = \varphi_t(e), \quad \text{where } \{\varphi_t\}$$

is the 1-parameter group assoc. to X.

This defines a map
$$\mathfrak{g} \xrightarrow{\exp} G$$
$$X \longmapsto \exp(X),$$

called the exponential map of G.

For $G = GL_n(\mathbb{R})$, we have seen that $\mathfrak{g} = (M_n(\mathbb{R}), [\,,\,])$, where $A \in M_n(\mathbb{R})$ corresponds to the vector field $\tilde{A} = \sum_{i,j} a_{ij} \dfrac{\partial}{\partial x_{ij}}$.

We then have the 1-parameter group $\{\varphi_t\}$ associated to $\tilde{A}$, and $\mathrm{Exp}_{\tilde{A}}(e) = \varphi_1(e)$

$\exp(\tilde{A}) =$ is the exponential of $\tilde{A}$. While

$$\cancel{\varphi_t(\tilde{A})} \quad \exp(t\tilde{A}) = \varphi_t(e).$$

Consider the curves $\gamma : \{t \mapsto e^{tA}\}$ on $GL_n(\mathbb{R})$.

By def^n of the maps $t \mapsto \exp(t\tilde{A})$, we have $\exp(t\tilde{A})$ is the maximal int. curve of $\tilde{A}$ through I, while γ satisfies

$$\dot{\gamma}(0)(f) = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma) \qquad (f \in C^\infty(GL_n))$$

$$= \frac{d}{dt}\Big|_0 f(e^{tA}) = \sum \frac{\partial f}{\partial x_{ij}}\Big|_I (A)_{ij}$$

$$= \left(\sum a_{ij} \frac{\partial}{\partial x_{ij}}\Big|_I\right)(f).$$

$$= \tilde{A}(I)(f).$$

Hence γ is the max int. curve of $\tilde{A}$ through I. Thus $\exp(tA) = e^{tA}$ $\forall\, A \in M_n(\mathbb{R})$. $\boxtimes$

By a similar computation (use left-inv.) to prove

<u>Lemma</u>:- For a Lie group G, $g \in G$ and $X \in \mathfrak{g}$, we have $\mathrm{Exp}_X(g) = L_g(\exp(X))$.

<u>Corollary</u>:- For $X \in \mathfrak{g}$, the flow of X is given by $F_X(t, g) = g\exp(tX)$.

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