Lecture - 15

Some recalls :→ We are studying vector fields on a smooth manifold M. We defined a smooth vector field as a map $X : M \to T(M) = \bigcup_p T_pM$ such that $X(p) \in T_pM$ $\forall$ $p \in M$ and on any local chart $(U, \varphi) \ni p$ of M, $X(q) = \sum_i f_i(q) \frac{\partial}{\partial x_i}\Big|_q$, $q \in U$; all f_i are smooth wrt any chart.

→ The set of all smooth vector fields on M is a real vector space $\mathcal{X}(M)$.

→ For $X, Y \in \mathcal{X}(M)$, (thinking of X as a derivation : $C^\infty(M) \to C^\infty(M)$; $X(f)(p) := X(p)(f)$)
$[X, Y] := X \circ Y - Y \circ X \in Der_{\mathbb{R}}(C^\infty(M), C^\infty(M))$ is a vector field, $(\mathcal{X}(M), [\,,\,])$ is a Lie algebra.

Rem: • In general $X \circ Y$ is not a vector field as this may involve higher order diff. operators

• For $X \in \mathcal{X}(M)$, $f \in C^\infty(M)$, $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$
$\equiv \mathcal{L}_X(f)(p) := X(p)(f)$; $\mathcal{L}_X(f)$ is called the Lie derivative of f wrt X.

We can think of $\mathcal{L}_X(f)$ as the function giving, for $p \in M$, the 'directional derivative' of f

at p, in the direction of $X(p) \in T_pM$.

→ $\mathfrak{X}(M)$ is a $C^\infty(M)$-module. For $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, $(fX)(p) := X(p)(f)$ defines the module structure.

→ $[\ ,\ ] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is only $\mathbb{R}$-bilinear, <u>not</u> $C^\infty(M)$-bilinear.*

→ $\mathcal{L}_X(Y) := [X, Y]$ is the Lie derivative of Y wrt X.

$\boxed{\begin{aligned} * \ [fX, gY] &= fg[X,Y] + f(Xg)Y \\ &\quad - g(Yf)X, \\ f, g &\in C^\infty(M). \end{aligned}}$

→ M, N smooth, $\varphi : M \to N$ smooth; $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ are φ-related if $\forall\, p \in M$,

$D\varphi(p)(X(p)) = Y(\varphi(p))$.

• X, Y are φ-related $\iff X(g \circ \varphi) = Y(g) \circ \varphi$ $\forall\, g \in C^\infty(N)$.

$\boxed{\begin{aligned} &M \xrightarrow{\varphi} N, \quad D\varphi(p) : T_pM \to T_{\varphi(p)}N \\ &D\varphi(p)(X(p))(g) = X(p)(g \circ \varphi) \\ &\quad \equiv Y(\varphi(p)) \quad \forall\, p. \end{aligned}}$

<u>Lie groups</u>: • Here the important notion is of left inv. vector fields. If G be a Lie grp $X \in \mathfrak{X}(G)$ is left-inv. if $DL_g \circ X = X \circ L_g$ $\forall\, g \in G$.

• Hence left-inv. vect. fields on G are completely determined by their value at $e \in G$.

15-3 → If X, Y are left inv. vector fields on G, then $[X, Y]$ is left-inv. Hence the subspace $\mathfrak{g}$ of $\mathfrak{X}(G)$ is a Lie sub-algebra. The map

$$T_eG \to \mathfrak{g}, \quad v \longmapsto X_v : \cancel{X_v(g) \in T_gG} \in \mathfrak{g}$$

is given by $X_v(g) := DL_g(e)(v) \in T_gG$.

→ We identify $\mathfrak{g}$ with T_eG via this isomorphism of $\mathbb{R}$-vector spaces and give T_eG a Lie alg. structure via this, i.e.

$$[v, w] := [X_v, X_w].$$

- <u>Integral curves</u>, <u>flows</u> of <u>vect. fields</u> :-

Given $X \in \mathfrak{X}(M)$ & $p \in M$, $\exists$ $\epsilon > 0$ and a curve $\gamma : (-\epsilon, \epsilon) \to M$ s.t. $\gamma(0) = p$, $\dot{\gamma}(0) \in T_pM$. and $X(p) = \dot{\gamma}(0)$. We call such a curve an integral curve of X through p, if $\forall t \in (-\epsilon, \epsilon)$, we have, in addition, $X(\gamma(t)) = \dot{\gamma}(t)$.

→ These exist. An integral curve through p having the largest possible interval of def^n is unique & is called the max int curve of X through p, denoted $\gamma_p : I_p \to M$.

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→ We call X __complete__ if __all__ integral curves of X have domain $= \mathbb{R}$.

- Such fields are important since they generate a global action of $\mathbb{R}$ on M

$$\mathbb{R} \times M \to M \quad , (t, p) \mapsto \gamma_p(t).$$

The map $\Phi_X : \mathbb{R} \to \mathrm{Diff}(M)$, $\Phi_X(t) = (p \mapsto \gamma_p(t))$ corresponds to this action.

- Writing φ_t for $\Phi_X(t)$, we see

$$\varphi_s \circ \varphi_t = \varphi_{s+t} \quad \forall\ s, t \in \mathbb{R}.$$

__Flows__ :- Given $X \in \mathfrak{X}(M)$, $p \in M$; $\exists\ \epsilon > 0$ and $p \in U \underset{\text{open}}{\subseteq} M$, a ! smooth map

$$F : (-\epsilon, \epsilon) \times U \to M \quad \text{with } F(0, x) = x$$

$$\forall\ x \in U \ ; \ \frac{d}{dt}(F(x,t)) = X(F(x,t)) \ \forall\ t \in (-\epsilon, \epsilon),\ x \in U.$$

So the existence & uniqueness theorem gives a ! family of integral curves of X on a nbd U of p, having the __same__ domain $I = (-\epsilon, \epsilon)$; $\gamma_p(t) = F(t, p) \neq \cancel{\varphi_t(p)}$.

• X <u>complete</u> gives $\{\varphi_t\}_{t\in\mathbb{R}}$. This is called the 1-param. group of diffeoms assoc. to X; $\varphi_t(p) := \gamma_p(t) = F(t,p)$.

When X is not nec. complete, we still get a 1-param. family of local diffeoms:

→ A local 1-param family of local diffeoms is a collection $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)\}_\alpha$ s.t.

(i) U_α are open, (ii) $\bigcup_\alpha U_\alpha = M$

(iii) $\epsilon_\alpha > 0 \ \forall \alpha$; domain $\varphi_t^\alpha \supseteq U_\alpha$ and the map $(t,z) \mapsto \varphi_t^\alpha(z)$ is smooth, for $z \in U_\alpha$, $t \in (-\epsilon_\alpha, \epsilon_\alpha)$.

(iv) φ_t^α & φ_t^β agree on $\mathrm{dom}(\varphi_t^\alpha) \cap \mathrm{dom}(\varphi_t^\beta)$ & t with $|t| < \min\{\epsilon_\alpha, \epsilon_\beta\}$.

(v) for $t, s \in (-\epsilon_\alpha, \epsilon_\alpha)$ with $t+s \in (-\epsilon_\alpha, \epsilon_\alpha)$, $\varphi_t^\alpha \circ \varphi_s^\alpha$ is defined on an open set $\supseteq U_\alpha$

& $\varphi_t^\alpha \circ \varphi_s^\alpha = \varphi_{t+s}^\alpha$ on U_α.

→ Any local 1-param. family is generated by a vector field, i.e. its local flow gives the 1-param. family: $X(p)(f) = \frac{d}{dt}\big|_{t=0} f \circ \varphi_t^\alpha$, $p \in M$.

<u>Sufficient cond^n for completeness</u>: Let $X \in \mathfrak{X}(M)$ be s.t. it admits a local 1-param family of diffeoms $\{(U_\alpha, \varphi_t^\alpha, \epsilon_\alpha)_\alpha\}$ with $\inf_\alpha(\epsilon_\alpha) > 0$. Then X is complete.

$\Rightarrow$ • Vector fields on M are complete if M is compact

• G a Lie group, $X \in \mathfrak{g}$, then X is complete.

<u>Exponential map</u>:- Let M be smooth, $X \in \mathfrak{X}(M)$ be complete & $\{\varphi_t\}$ be the 1-param group assoc. to X. The <u>exponential map</u> of X is $\mathrm{Exp}_X : M \to M$, $\mathrm{Exp}_X(p) := \varphi_1(p) = \gamma_p(1)$.

<u>Examples</u>:- 1. $A \in M_n(\mathbb{R})$, $A = (a_{ij})$. Let

$$\hat{A} \in \mathfrak{X}(\mathbb{R}^n), \quad \hat{A} = \sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_i}.$$

Then $\mathrm{Exp}_{\hat{A}}(p) = e^A \cdot p$ (see lect. 14)

2. If G be a Lie group, <u>any</u> $X \in \mathfrak{g}$ is complete, Exp_X is thus defined. Define

$$\exp(X) := \mathrm{Exp}_X(e) = \varphi_1(e) = \gamma_e(1),$$

$$\exp(tX) = \varphi_t(e), \quad \{\varphi_t\} \text{ is the 1-param}$$

group assoc. to X. The map $\mathfrak{g} \xrightarrow{\exp} G$

15-7 is called the exponential map of G.

→ $\mathrm{Exp}_X(g) = L_g(\exp(X)), \ \forall g \in G.$ (*)

For $G = GL_n(\mathbb{R})$, $\exp(tA) = e^{tA}$.

→ For $X \in \mathfrak{g}$, the flow of X is given by $F_X(t,g) = g\exp(tX)$.

- If G is a Lie group, then the map $\mathbb{R} \to G$, $t \mapsto \exp(tX)$ is the ! maximal integral curve of X, passing through e.

<u>Prf</u>:- We have, for $g \in G$, $\varphi_t(g) = \gamma_g(t)$

$$= F_X(t,g) = g\exp(tX).$$

Hence $t \mapsto \exp(tX)$ is the (!) max integral curve of X through e. ⊠

<u>Some properties of exp:</u>

<u>Lemma</u>: $\exp((s+t)X) = \exp(sX)\cdot\exp(tX)$ $\forall$ $s,t \in \mathbb{R}$, $X \in \mathfrak{g}$.

<u>Proof</u>:- For $g \in G$, $X \in \mathfrak{g}$, $t \in \mathbb{R}$, $h \in G$

$$(L_g \mathrm{Exp}_{tX} L_{g^{-1}})(h) = L_g \mathrm{Exp}_{tX}(g^{-1}h) = L_g L_{g^{-1}h}(\exp(tX)) = L_h(\exp tX) = \mathrm{Exp}_{tX}(h)$$

(by (*)). Hence $L_g \mathrm{Exp}_{tX} = \mathrm{Exp}_{tX} L_g$.

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~~Now~~ We have, for $g, h \in G$,

$\cancel{\exp((s+t)X)} =$

$(L_g \operatorname{Exp}_{tX} L_{g^{-1}})(h) = L_g \operatorname{Exp}_{tX}(g^{-1}h) = L_g L_{g^{-1}h}(\exp(tX))$

by $(*)$.

$= L_h \exp(tX)$

$= \operatorname{Exp}_{tX}(h)$

$\therefore \quad L_g \operatorname{Exp}_{tX} = \operatorname{Exp}_{tX} L_g.$

$\rightarrow \quad \exp((s+t)X) = \varphi_{s+t}(e) = \varphi_s(\varphi_t(e))$. And

$\varphi_s(\varphi_t(e)) = \cancel{\varphi_s(L_{\varphi_t(e)}(e))} = \gamma_{\varphi_t(e)}(s)$

$= \exp((t+s)X)$

$= \varphi_t(e) \underbrace{\exp(sX)}_{=\varphi_s(e)}$ by $(*)$

$= \varphi_t(e)\, \varphi_s(e)$

$= \exp(tX) \exp(sX)$. $\boxtimes$

$\therefore \quad \exp : \mathbb{R} \to G$ is a Lie group homom.

- Any Lie group map ~~&~~ $\gamma : \mathbb{R} \to G$ is also called a 1-parameter subgroup of G.

$\rightarrow \quad \exp : \mathfrak{g} \to G$ is a local diffeom around $0 \in \mathfrak{g}$. [See Kum; 2.8.28, follows from exp is smooth]

159. Grant $\exp : \mathfrak{g} \to G$ is smooth. Enough to prove $D(\exp)(0)$ is bijective. Let $X \in \mathfrak{g} \equiv T_e G$. Consider the curve $t \mapsto tX$ in $\mathfrak{g}$. This has tangent X at 0. Consider the image of this curve under exp, i.e. $t \mapsto \exp(tX)$; this has tangent X at e. This shows $D(\exp)(0) : \mathfrak{g} \to T_e G$ is bijective etc. $\boxtimes$

→ Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\varphi"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

commutes, i.e. $\exp(d\varphi(X)) = \varphi(\exp(X))$ $\forall X \in \mathfrak{g}$.

Pf: Consider the curves $t \mapsto \varphi(\exp(tX))$ & $t \mapsto \exp t(d\varphi(X))$. These are both integral curves for $d\varphi(X)$, through e $\therefore$ are equal. $\boxtimes$

→ Any one parameter subgroup $\varphi : \mathbb{R} \to G$ comes from some $X \in \mathfrak{g}$. More precisely, $\exists$ $X \in \mathfrak{g}$ s.t. $\varphi(t) = \mathrm{Exp}_{tX}(e)$.

B. Math (Hons.) III Year 2021-22

II Semester

ATTENDANCE SHEET

Course : Differential Geometry II

Instructor : Maneesh Thakur

SL.	Name	March												April																Total Class
	DAY	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	Tue	Tue	Thu	Thu	
	DATE	15	15	17	17	22	22	24	24	29	29	31	31	5	5	7	7	12	12	14	14	19	19	21	21	26	26	28	28	28
1	P Anirudha Chakravarthi																													
2	Anupam Nayak																													
3	Arun Vittal Wandakar																													
4	Debabrata Pal																													
5	Harshul Khanna																													
6	Kuntal Das																													
7	Muhammed K M																													
8	Parijat Chakraborty																			Dr. Ambedkar Jayanti / Mahavir Jayanti										
9	Rashmi Konnur																													
10	Rishika Agrawal																													
11	Rohan Raj Padhy																													
12	Safal Das Biswas																													
13	Sayantan Mondal																													
14	Shahbaz Ahmad Khan																													
15	Shubhankar Varshney																													
16	Sneha K K																													
17	Soham Ghosh																													
18	Sonali Priyadarsini Behara																													

Instrutor Signature :