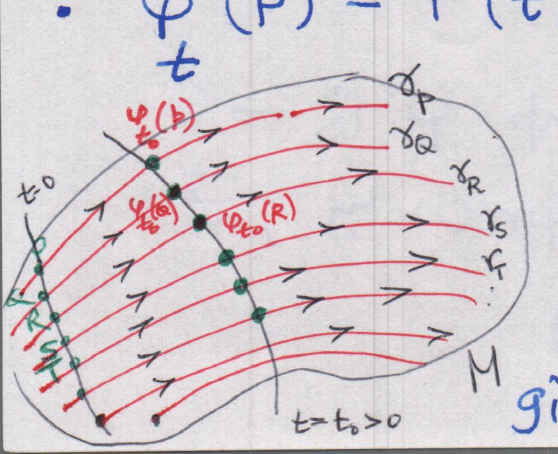
Lecture-16

Exponential map: We discussed complete vector fields on manifolds. These are vector fields X on M whose all integral curves (maximal ones) have domain $\mathbb{R}$. If X be such a vector field, F its flow on M; $F: \mathbb{R}\times M \to M$ smooth with • $F(0,x) = x$ $\forall x \in M$, • $\frac{d}{dt} F(t,x) = X(F(t,x))$ $\forall t \in \mathbb{R}, x \in M$. This gives $\phi: \mathbb{R} \to \mathrm{Diff}(M)$ a group homomorphism, $\phi(t) =: \varphi_t$, i.e. $\varphi(t)(p) := F(t,p)$

$\varphi_s \circ \varphi_t = \varphi_{s+t}$ $\forall s,t \in \mathbb{R}$, $\varphi_0 = \mathrm{Id}_M$. The set $\{\varphi_t\}_{t\in\mathbb{R}}$ is thus a group under $\circ$, $\varphi_t^{-1} = \varphi_{-t}$; called the 1-parameter group of diffeoms assoc. to X. A few remarks are in order:

- The flow of X gives the set $\{\gamma_p\}_{p\in M}$ of max integral curves of X, all having domain $\mathbb{R}$.
- $\varphi_t(p) = F(t,p) = \gamma_p(t)$; $\dot{\gamma}_p(t) = X(\gamma(t))$ $\forall t \in \mathbb{R}$.



- The 1-param group $\{\varphi_t\}$ moves or translates the points of M simultaneously along the max integral curves through those points, as in the picture, giving the action of $\mathbb{R}$ on M.

Now let $X \in \mathcal{X}(M)$ be complete, $\{\varphi_t\}$ the corresp. 1-param group. Define the exponential map of X by $\mathrm{Exp}_X : M \to M$, $\mathrm{Exp}_X(p) = \varphi_1(p) = \gamma_p(1)$, $p \in M$. So $\mathrm{Exp}_X(p)$ is the point on the max int. curve γ_p at $t=1$. We discuss one case of M in some detail, when we know any left inv. vector field is complete, $M = G =$ a Lie group.

We'll discuss (next Tuesday afternoon)

*Exercise 1. $\mathrm{Exp}_X(g) = L_g(\exp(X))$, where

$$\exp(X) := \mathrm{Exp}_X(e) = \varphi_1(e) \text{ is the map}$$

$\mathfrak{g} \equiv T_eG \to G$, called the exponential map of G. $\left(\exp(tX) := \varphi_t(e)\right) = \gamma_e(t)$

- For $X \in \mathfrak{g}$, the flow F_X of X is given by $F_X(t,x) = x\exp(tX)$ for any $t \in \mathbb{R}$, $x \in G$.
- $t \mapsto \exp(tX)$ is the max integral curve of X through e. (*~~Exercise 2~~)

~~This follows from the following computation~~

~~*Exercise 2:~~ $\cancel{\mathrm{Exp}_{tX}(e) = \varphi_t(e)}$

- $\exp((s+t)X) = \exp(sX)\exp(tX)$ $\forall s,t \in \mathbb{R}$, $X \in \mathfrak{g}$.

This follows from
$$L_g \, \mathrm{Exp}_{tX} = \mathrm{Exp}_{tX} \, L_g \, . \qquad (*)$$

$$\exp((s+t)X) = \varphi_{s+t}(e) = \varphi_s(\varphi_t(e))$$
$$= \varphi_t(\varphi_s(e)) = \varphi_t(L_{\varphi_s(e)}(e))$$
$$= L_{\varphi_s(e)}(\varphi_t(e))$$
$$= \varphi_s(e)\,\varphi_t(e) \, .$$

(Left inv. of $X \equiv DL_g \circ X = X \circ L_g$ is used to have $(*)$).

<u>Exercise</u>: Fill details in the above.

- $\exp : \mathfrak{g} \to G$ is a local diff-co around $0 \in \mathfrak{g}$.

To prove this, we think of $\mathfrak{g}$ as a Euclidean space, prove • exp is smooth : $\mathfrak{g} \to G$ and observe, for $X \in \mathfrak{g}$, the curve $t \mapsto tX$ in $\mathfrak{g}$ has X as its tangent at 0 & exp takes this curve to the curve $t \mapsto \exp(tX)$, whose tangent at $t=0$ is $X(e)$.

 We'll need

Lemma: $\phi : G \to H$ be a homom of Lie groups,

then
$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\phi"', "D\phi(e)"] \arrow[ur, phantom, "\circlearrowright"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$
i.e. $\exp(d\phi(x)) = \phi(\exp(x))$ $\forall\ x \in \mathfrak{g}$.

Pf: Compare the curves $t \mapsto \phi(\exp(tx))$ & $t \mapsto \exp(t\,d\phi(x))$ on H through $e \in H$; both are integral curves of $d\phi(x)$ (?). $\boxtimes$

Cor: • $d\phi(x) = \frac{d}{dt}\phi(\exp(tx))\Big|_{t=0}$.

• $\phi, \psi : G \to H$ be Lie group homoms, G connected and $d\phi = d\psi$. Then $\phi = \psi$.

Proof: We have, for all $x \in \mathfrak{g}$
$$\exp(d\phi(x)) = \phi(\exp(x)) = \exp(d\psi(x)) = \psi(\exp(x)).$$
Since $\exp : \mathfrak{g} \to G$ is a local diffeo, we conclude that $\varphi = \psi$ on an open nbd of $e \in G$. The result follows from this (?). $\boxtimes$

Exercises: Let G be a topological group, i.e. G is Hausdorff, $G \times G \xrightarrow{\mu} G$, $i: G \to G$ are continuous.

1. Let U be an open set $\ni e$. Then $\exists\ e \in V \underset{\text{open}}{\subset} U$ s.t. $V = V^{-1}$ & $VV = VV^{-1} \subseteq U$.
(Such open sets V are called symmetric)

Hint: Use contin. of μ to find opens P, Q both $\ni e$ s.t. $P \cdot Q \subseteq U$, $W = P \cap Q$ & take $V = W \cap W^{-1}$.

2. $G = \bigcup_{k=1}^{\infty} U^k := \{u_1 \cdots u_k \mid u_i \in U\}$. if G is conn. & U ~~symmetric~~ any open $\ni e$. So U generates G as an abstract group.

Frobenius theorem & applications

Let M be smooth, $X \in \mathfrak{X}(M)$ be s.t. $X(p) \neq 0$ $\forall\ p \in M$. We have seen then that $\exists$ integral curve γ_p of X through p. The condition on $X \Rightarrow \dot{\gamma}_p(0) \neq 0$. This implies, for $\epsilon > 0$ suff. small, $\gamma_p : (-\epsilon, \epsilon) \to M$ is a 1-1 smooth immersion (?).

• The cond^n $X(p) \neq 0$ $\forall\ p \in M$ implies that $\mathcal{D}_p := \langle X(p) \rangle \subset T_pM$ is a 1-dim'l subspace $\forall p$.

16-6 We call the association $p \mapsto \mathcal{D}_p$ a 1-dim'l "differential system" on M.

So, the existence of integral curves of X can be rephrased by saying: given a 1-dim'l differential system on M and $p \in M$, $\exists$ a submanifold $S \subseteq M$, $p \in S$, such that $T_q S = \mathcal{D}_q$ $\forall q \in S$, with $S = ((-\epsilon, \epsilon), \gamma_p)$.

More generally, we <u>define</u> a d-dim'l diff system on M, $p \mapsto \mathcal{D}_p \subset T_p M$, $\mathcal{D}_p$ a d dim'l subspace of $T_p M$; we also demand

- $\forall p \in M$, $\exists p \in \underset{\text{open}}{U \subseteq M}$, $X_1, \dots, X_d \in \mathfrak{X}(U)$

such that $\mathcal{D}_q = \langle X_1(q), \dots, X_d(q) \rangle$ $\forall q \in U$.

One then says <u>the vector fields $X_1, \dots, X_d$ span $\mathcal{D}$</u> & call $\mathcal{D}$ <u>smooth</u>.

<u>Question</u>:- Given a smooth d-dim'l diff. system on M, $p \in M$, ? $\exists$ a submanifold S passing through p & $T_q S = \mathcal{D}_q$ $\forall q \in S$?.

We define a few notions first:

<u>Involutive diff systems</u>:- A diff. system $\mathcal{D}$ is <u>involutive</u> if $X, Y \in \mathcal{D} \Rightarrow [X, Y] \in \mathcal{D}$, i.e.

$\longrightarrow$

for any $p \in M$, $[X,Y](p) \in \mathcal{D}_p$.

- We call a submanifold S of M an <u>integral</u> submanifold at p for $\mathcal{D}$ if $p \in S$ and $\forall\, q \in S$,

$$T_q S = \mathcal{D}_q .$$

- We call $\mathcal{D}$ <u>integrable</u> if for every $p \in M$, there is an integral submanifold through p.

<u>Frobenius theorem</u>:- A differential system $\mathcal{D}$ is integrable $\Longleftrightarrow$ $\mathcal{D}$ is involutive.

<u>Remarks/example</u>: • Let $\mathcal{D}$ be a d-dim'l diff system. Then for $p \in M$, $\exists\ \underset{p}{\underset{\cup}{U}} \subseteq M$ open, $X_1, \dots, X_d \in \mathcal{X}(U)$ s.t. $X_1, \dots, X_d$ span $\mathcal{D}$ on U.

To say $\mathcal{D}$ is involutive is equivalent to existence of functions c_{ijk} s.t., $\forall\, i,j$,

$$[X_i, X_j] = \sum_{k=1}^{d} c_{ijk} X_k .$$

<u>Example</u>:- The above is ensured for example when $\exists$ open U, $p \in U$, such that $\exists$ $X_1, \dots, X_d \in \mathcal{X}(U)$ with $[X_i, X_j] = 0\ \forall\, i,j$ & $X_i \in \mathcal{D}\ \forall\, i$. For a concrete example, take $X_i = \frac{\partial}{\partial x_i}$ on U

<u>The global Frobenius theorem</u> :- Let $\mathcal{D}$ be a d-dim'l diff. system, involutive, on M & $p \in M$. Then $\exists!$ maximal connected integral submanifold of $\mathcal{D}$, passing through p. Every connected integral submanifold of $\mathcal{D}$ through p, is contained in <u>the</u> maximal one.

Application to Lie groups

<u>Thm</u>: Let G be a Lie group, $\mathfrak{g} = \text{Lie}\, G$ & $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra. Then $\exists!$ connected Lie subgroup $H \leq G$ with $\text{Lie}(H) = \mathfrak{h}$.

<u>Prf</u>: ~~For~~ $g \in G$, ~~and~~ define $\mathcal{D}_g = \{X \in \mathfrak{g} \mid X(g) \in \mathfrak{h}\}$. Then $\mathcal{D}$ is involutive diff system of rank $d = \dim \mathfrak{h}$ for every $g \in G$.

By the Global Frobenius, $\exists!$ a maximal integral submanifold $H \ni e$, of $\mathcal{D}$.

<u>Claim</u>: $H \leq G$. To see this, let $h \in H$. Then $L_h(H)$ is a connected integral submfld for $\mathcal{D}$ & $h \in L_h(H)$; Hence $L_h(H) \subseteq H$. This implies $H \leq G$.

→ One can prove further, H is a Lie subgroup. $\boxtimes$

Hence we have (Frobenius' thm).

Theorem: G a Lie group, $\mathfrak{g} = \mathrm{Lie}\,G$. Then $\exists$ a bijective correspondence between connected Lie subgroups of G and Lie subalgs of $\mathfrak{g}$. $\boxtimes$

Some useful exponential formulae

Thm: Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}\,G$; $X, Y \in \mathfrak{g}$. Then

(i) $\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right)$

(ii) $\exp(tX)\exp(tY)\exp(-tX)\exp(-tY)$
$$= \exp\left(t^2[X,Y] + O(t^3)\right)$$

(iii) $\exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X,Y] + O(t^3)\right)$.

Where $O(t^3)$ is a vector in T_eG such that for $|t| < \epsilon$ $t \mapsto \left(\frac{1}{t^3}\right)O(t^3)$ is bounded & smooth. $\boxtimes$

Cartan's theorem:- G be a Lie group, $H \leq G$ a closed subgrp. Then $\exists!$ diff. structure on H so that H is a Lie group & the inclusion map $H \hookrightarrow G$ is a Lie homom. $\boxtimes$

If $H \leq G$ is a Lie subgroup, then
$$\mathrm{Lie}(H) = \{X \in \mathfrak{g} \mid \exp(tX) \in H \ \forall\, t \in \mathbb{R}\}.$$

<u>The adjoint representation</u> :- Let G be a Lie grp, $g, x \in G$, $\iota_g : G \to G$ be the map $x \mapsto gxg^{-1}$. Then ι_g is a Lie group homomorphism.

<u>Notation</u> : Let $Ad(g) = d\iota_g$ correspond to the derivative of $\iota_g : G \to G$, So $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ is an element of $GL(\mathfrak{g})$.

<u>Lemma for GL_n</u> :- Let $G = GL_n(\mathbb{R})$, $A \in GL_n$, So $\iota_A(X) = AXA^{-1}$. We have $Ad(A)(X) = AXA^{-1}$ $\forall\, X \in \mathfrak{gl}_n(\mathbb{R}) =: \mathfrak{g}$.

<u>Pf</u> :- Recall that $\exp : \mathfrak{gl}_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is the matrix exponential and $t \mapsto \exp(tX)$ is the integral curve through I for X.

Hence $Ad(A)(X) = d\iota_A(X) = \frac{d}{dt} \exp(tAXA^{-1})\Big|_{t=0}$

$\left(= \frac{d}{dt} A \exp(tX) A^{-1} \right)$. ~~Since~~ ~~exp is a local diffeom~~, it follows that

$Ad(A)(X) = AXA^{-1}$. $\boxed{\ast}$.

→ $Ad : GL_n \to \mathfrak{gl}_n$ is a Lie group homom, Called the adjoint representⁿ of G on $\mathfrak{g}$.