Lecture - 17

We discussed exponential map for Lie grps.

- G a Lie group, $\mathfrak{g}$ its Lie algebra, then $\exp: \mathfrak{g} \longrightarrow G$ is a local diffeom around 0.

This is an extremely useful result for various reasons.

<u>Proposition</u>:- Let $\varphi : G \to H$ be a map of Lie groups, then the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\varphi"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

commutes, so

$$\exp(d\varphi(x)) = \varphi(\exp(x)) \quad \forall\, x \in \mathfrak{g}.$$

<u>Proof</u>:- Consider the curves $t \mapsto \varphi(\exp(tx))$ and $t \mapsto \exp(t\, d\varphi(x))$ on H. Both these pass through $e \in H$, have domain $\mathbb{R}$, and their derivatives at $t=0$ give $d\varphi(x) \in \mathfrak{h}$, hence they must be equal; thus, at $t=1$, we get $\exp(d\varphi(x))$
$= \varphi(\exp(x))$. $\boxtimes$

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This equality gives, treating $t \mapsto \phi(\exp(tx))$ as the max curve of $d\varphi(x)$,

$$d\varphi(x) = \frac{d}{dt}\Big|_{t=0} \varphi(\exp(tx)).$$

Another imortant result; applic^{n} of the above,

Proposition :- $\phi, \psi : G \to H$ be Lie group homomorphisms with $d\varphi = d\psi$. Then $\varphi = \psi$ if we assume G is connected.

Proof :- We have, for any $X \in \mathfrak{g}$

$$\exp(d\varphi(x)) = \phi(\exp(x)) = \exp(d\psi(x))$$
$$\mathfrak{h} \longrightarrow H \qquad = \psi(\exp(x)).$$

Since $\exp : \mathfrak{g} \to G$ is a local diffeom, we conclude that $\varphi = \psi$ on a nbd of $e \in H$. The result now follows from

- In a connected topological group G, any nbd of e generates G
- φ & ψ are homomorphisms.

$\rightsquigarrow$

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Discussion on Frobenius Thm, sketch of a proof: Let M be a manifold and $\mathcal{D}$ be a d-dim'l diff. system ($\equiv$ distribution) on M, What does this mean?

- $\mathcal{D}$ assigns to each $p \in M$, a d-dim'l subspace $\mathcal{D}_p$ of T_pM
- For every $p \in M$, $\exists$ $U \underset{\text{open}}{\subseteq} M$, $p \in U$ and smooth vector fields $X_1, \ldots, X_d \in \mathfrak{X}(U)$ such that $\forall$ $q \in U$, (We then say $\mathcal{D}$ is smooth) "$\mathcal{D}$ is smooth"

$$\mathcal{D}_q = \langle X_1(q), \ldots, X_d(q) \rangle \quad \text{(lin. span)}$$

→ This generalizes the situation of a vector field on M which has no zeros on M. Given such a vector field, we have integral curves through points of M, that 'fit' the field.

Q: Given smooth $\mathcal{D}$ on M & $p \in M$, $\exists$? a manifold S & a smooth immersion/embedding $i: S \to M$, s.t. $p \in i(S)$ & $i_*: T_q(S) \to T_{i(q)}M$ has image $\mathcal{D}_{i(q)} \in \mathcal{D}$ $\forall$ $q \in S$.

We then call S an integral (sub)manifold of $\mathcal{D}$. We have a nec. condⁿ for existence for such S: first, recall that we say a vector field $X \in \mathcal{D}$ if $X(q) \in \mathcal{D}_q$ for any $q \in M$.

Now if $Y_1, Y_2 \in \mathfrak{X}(S)$, then for $i: S \to M$, $i_*(Y_1), i_*(Y_2) \in \mathcal{D}$,

$$[i_* Y_1, i_* Y_2] = i_*([Y_1, Y_2])$$

<u>This forces</u>: Whenever $X, Y \in \mathfrak{X}(M)$ & $X, Y \in \mathcal{D}$, then $[X, Y] \in \mathcal{D}$, if $\mathcal{D}$ were to admit an int. manifold.

Such $\mathcal{D}$ is called <u>integrable</u>.

<u>A non-example + exercise</u>: The 2-dim'l system spanned by $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y}$, is <u>not</u> integrable. Here X, Y define a 2-dim'l distribution on $\mathbb{R}^3$.

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We call a distribution $\mathcal{D}$ involutive, if

$X, Y \in \mathcal{D} \Rightarrow [X, Y] \in \mathcal{D}$

(Note that $\mathcal{D}$ is <u>not</u> a set of vect. fields!)

<u>Frobenius thm</u> (local): Let M be a manifold, $p \in M$. Then a d-dim'l distribution is integrable around p ($\equiv \exists$ an integral submfld passing through p) if and only if $\mathcal{D}$ is involutive.

Let us discuss this. The system on $\underset{M:=}{\mathbb{R}^3} - \{x=y=0\}$ spanned by $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial z}$ is involutive, since $[X, Y] = 0$;

<u>Integral curves</u>: For $(x, y, z) \in M$, the integral curve is a circle in the z-plane, with center at the origin, the integral curve of Y is the line $\parallel$ to the z-axis. The integral manifold through (x, y, z) is the cylinder with z axis as its central axis (Exercise: verify this)

17-6: If $\mathcal{D} = \langle X_1, \dots, X_d \rangle$ on $U \ni p$, then $\forall\ i,j$ we can write

$$[X_i, X_j] = \sum_{k=1}^{d} c_{ijk} X_k \quad , \ c_{ijk} \text{ smooth on } U;$$

assuming $\mathcal{D}$ is involutive. One obvious case of this is when all X_i's commute i.e. $c_{ijk} = 0 \ \forall\ i,j,k$, so $[X_i, X_j] = 0 \ \forall\ i,j$. There are a couple of technical lemmas that go into helping us:

1. Let $X \in \mathcal{X}(M)$, $p \in M$ with $X(p) \neq 0$. Then $\exists$ a coordinate chart $U \ni p$ such that $X = \frac{\partial}{\partial x_1}$ on U. (Thm 2.9.2, Kum)

2. $X_1, \dots, X_d \in \mathcal{X}(M)$, $p \in M$, be lin. indep on $U \ni p$ (i.e. $\langle X_1(q), \dots, X_d(q) \rangle$ is d-dim'l $\forall\ q \in U$). Assume $[X_i, X_j] = 0 \ \forall\ i,j$. Then $\exists$ a coord chart $V \ni p$ s.t. $X_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq d$, on V. $\boxtimes$

So, by (2), $\exists$ local coordinates s.t.

$X_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, d$; $x_i(p) = 0$, $1 \leq i \leq m = \dim M$.

The submanifold S through p ~~has~~ should have

$$\langle X_1(p), \ldots, X_d(p) \rangle = T_p S = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right\rangle.$$

S is the 'slice'

$$\{ q \in U \mid x_j(q) = 0,\ d+1 \leq j \leq m \}$$

and for any $q \in U$, the integral submfld of $\mathcal{D}$ through q is given by the slice

$$\{ z \in U \mid x_j(z) = x_j(q),\ j \geq d+1 \}.$$

The general case is reduced to the above: Write $X_i = \sum_j f_{ij} \frac{\partial}{\partial x_j}$, $f_{ij} \in C^\infty(U)$. The matrix $(f_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq m}}$ then has rank d.

Wlg can assume $(f_{ij})_{\substack{1 \leq i \leq d, \\ 1 \leq j \leq d}}$ is invertible in a smaller nbd $V \subseteq U$ of p; let $(g_{ij})_{1 \leq i,j \leq d}$ be its inverse. Define

$$Y_i = \sum g_{ij} X_j.$$

Then $Y_i \in \mathcal{D}$ $\forall i$, $\langle Y_1, \ldots, Y_d \rangle = \mathcal{D}$, $[Y_i, Y_j] = 0$ $\forall i,j$.

$\boxtimes$