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Lecture - 18

Discussion on Frobenius theorem :- We'll discuss mainly examples & notions. Let M be an m-dim'l manifold. We then saw that any smooth vector field X on M can be 'integrated' locally around any point, to an integral curve. • If $X(p)=0$, then the corresp. integral curve is the constant curve $\gamma_p(t)=p$ $\forall\, t$. While $X(p)\neq 0$, then the integral curve $\gamma_p(t)$ is a 1-dim'l curve near p.

Higher dim'l analogue :- A d-dim'l distribution $\mathcal{D}$ on M is a map (?) that assigns to every $p\in M$ a d-dim'l subspace $\mathcal{D}_p$ of T_pM.

We call $\mathcal{D}$ smooth if for every $p\in M$, $\exists$ a nbd U of p & $X_1,\ldots,X_d\in\mathfrak{X}(U)$ s.t. $\forall\, q\in U$, $\{X_1(q),\ldots,X_d(q)\}$ is a basis of $\mathcal{D}_p$; (in particular, $X_1(q)\neq 0,\ldots,X_d(q)\neq 0$ $\forall\, q\in U$).

We'll discuss only smooth distributions.

• We say a vector field X belongs to $\mathcal{D}$ if

$X(p) \in \mathcal{D}_p \ \forall\, p \in M$.

(in the language of bundles, $\mathcal{D}$ is a rank-d subbundle of TM).

<u>Integral manifolds</u>:- Let $\mathcal{D}$ be a d-dim'l distribution ($\equiv$ diff. system) on M. $p \in M$. A natural question is: does $\exists$ a manifold S and a smooth 1-1 immersion $\iota : S \to M$ s.t. $p \in \iota(S)$ and $\iota_* = D\iota(q) : T_q(S) \to T_{\iota(q)}M$ has image $\mathcal{D}_{\iota(q)}$ $\forall\, q \in S$.

If such (S, ι) exists, we call it an <u>integral manifold of $\mathcal{D}$</u>. We call $\mathcal{D}$ <u>integrable</u> if through each $p \in M$, $\mathcal{D}$ admits an integral manifold.

<u>Examples</u>:- 1. Let X be a non-vanishing vector field on M. Then $p \overset{\mathcal{D}}{\longmapsto} \langle X(p) \rangle$ is clearly a 1-dim'l distribution on M. Any integral curve of X is an integral manifold of $\mathcal{D}$.

2. The vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$

span a k-dim'l (smooth) distribution $\mathcal{D}$ on $\mathbb{R}^n$.
The integral manifolds of $\mathcal{D}$ are the 'planes' defined by the system of eq^{n}s

$$x_j = c_j \ , \ k+1 \leq j \leq n.$$

3. Let $M = S^1 \times S^1 \subset (\mathbb{R}^2, \{x\}) \times \{\mathbb{R}^2, \{y\}\}$ and c be a fixed irrational $\in \mathbb{R}$. The non-vanishing vector field

$$X = \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right) + c \left(y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} \right)$$

defines a 1-dim'l distribution on M, the integral manifold is a dense curve in M!

<u>Remark</u>: Though every 1-dim'l distribution is integrable, the same <u>fails to be true for higher dim'l distributions</u>.

<u>Example</u>:- Consider the 2-dim'l distribution $\mathcal{D}$ on $\mathbb{R}^3$ defined by (spanned by) the vector fields $X_1 := \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$ and $X_2 := \frac{\partial}{\partial y}$, $\{x, y, z\}$ usual coords on $\mathbb{R}^3$.

Then $\mathcal{D}$ is <u>not</u> integrable. In fact, there is <u>no</u> integral manifold through $(0,0,0)$.

(should discuss this).

Observe that $[X_1, X_2] = -\frac{\partial}{\partial z} \notin \mathcal{D}$.

- Let us discuss <u>conditions for integrability</u>:

If (S, ι) is an integral manifold for $\mathcal{D}$, and $Y_1, Y_2 \in \mathfrak{X}(S)$, then $\iota_*(Y_1), \iota_*(Y_2) \in \mathcal{D}$ (?); $[Y_1, Y_2] \in \mathfrak{X}(S)$, so

$$\iota_*([Y_1, Y_2]) = [\iota_*(Y_1), \iota_*(Y_2)] \in \mathcal{D}.$$

This therefore gives us a nec. cond^n for $\mathcal{D}$ to admit an integrable manifold:

- We call a distribution $\mathcal{D}$ <u>involutive</u> if $X, Y \in \mathcal{D} \Rightarrow [X, Y] \in \mathcal{D}$.

- The above example is <u>not</u> involutive, hence not integrable!

<u>Examples</u> (Involutive dist.) • Any 1-dim'l distribution is involutive, since $[fX, gX]$
$= f(X(g))X - g(X(f))X$.

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- Let $f: M \to N$ be a submersion. Define the distribution $\mathcal{D}$ by $\mathcal{D}_p = \operatorname{Ker}(Df(p)) \subseteq T_pM$

If X, Y be vector fields belonging to $\mathcal{D}$, then $Df(p)(X(p)) = Df(p)(Y(p)) = 0$. Hence both X, Y are f-related to $0 \in \mathfrak{X}(N)$.

Clearly $Df(p)([X,Y](p)) = 0$. Hence $\mathcal{D}$ is involutive.

$\to$ The integral manifold passing through $p \in M$ is the submanifold $f^{-1}(f(p))$.

Take up Lecture-17, page-3 onwards from here.
