Lecture-19

→ Discuss proof of local Frobenius (Lect-17)
→ Global Frobenius w/o proof.
→ Applic^n to Lie groups – Lie alg. correspondence.

Lie groups (contd). We briefly discuss the adjoint rep^n of a Lie group on its Lie alg. The general philosophy is that 'to study a group, one must write down all it's rep^{n}s'. The adjoint rep^n is natural to the structure of the Lie group and is realized concretely as a rep^n. We proved, for the conjugation $\iota_g : G \to G$, $D(\iota_g)(e) : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra autom, so $D(\iota_g)(e) =: Ad(g) \in \mathfrak{gl}(\mathfrak{g})^{=GL(\mathfrak{g})}$, and we get the adjoint rep^n $Ad : G \to GL(\mathfrak{g})$. When $G = GL_n$, $A \in GL_n$, then $Ad(A)(X) = AXA^{-1}$, $\forall\, X \in \mathfrak{gl}_n = Lie(GL_n)$. We have, $Ad : G \to GL(\mathfrak{g})$ is smooth, hence $D(Ad)(e) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra map. We now compute this.

↝

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Example : $G = GL_n$, $\mathfrak{g} = \mathfrak{gl}_n = M_n(\mathbb{R})$.

Denoting $D Ad(e) =: ad$, we claim $ad : \mathfrak{g} \to \mathfrak{gl}_n$ is given by

$ad(X)(Y) := XY - YX = [X, Y]$.

We are looking to differentiate $Ad : GL_n \to GL(\mathfrak{gl}_n)$.

Using the commutative diag

$$\begin{array}{ccc} GL_n & \xrightarrow{Ad} & GL(\mathfrak{gl}_n) \\ \uparrow \exp & \circlearrowleft & \uparrow \exp \\ \mathfrak{gl}_n & \xrightarrow{ad} & \mathfrak{gl}(\mathfrak{gl}_n) \end{array}$$

we get,

using $\exp(X) = e^X$ for any matrix X,

$e^{ad(X)} = Ad(e^X)$. For $t \in \mathbb{R}$, this gives

$e^{ad(tX)} = Ad(e^{tX})$. Hence, for any $Y \in \mathfrak{gl}_n$, we get

$\cancel{e^{ad(tX)}(tY)e^{ad(-tX)}} =$

$\left(I + ad(tX) + ad(tX)^2 + \cdots\right)(tY)$

$= e^{tX} \cdot tY \cdot e^{-tX}$.

$LHS = tY + t^2 ad(X)(Y) + t^3 ad(X)^2(Y) + \cdots$

$\to$

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RHS

$$= \left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots\right) tY \left(1 - tX + \frac{t^2X^2}{2!} - \frac{t^3X^3}{3!} + \cdots\right)$$

$$= \left(tY + t^2XY + \frac{t^3X^2Y}{2!} + \frac{t^4X^3Y}{3!} + \cdots\right)\left(\quad \right)$$

$$= \underbrace{tY}_{\text{deg 1 in } t} + \underbrace{(-t^2YX + t^2XY)}_{\text{deg 2 in } t} + \underbrace{(\;\cdots\;)}_{\text{deg} \geq 3 \text{ in } t}$$

Comparing the two, we get

$$\mathrm{ad}(X)(Y) = XY - YX = [X,Y].$$

The general case is similar:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"] \arrow[ur, phantom, "\circlearrowleft"] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd} \implies \exp(\mathrm{ad}(X)) = \mathrm{Ad}(\exp(X)) \quad \forall X \in \mathfrak{g}.$$

So, for $t \in \mathbb{R}$ (small) & $Y \in \mathfrak{g}$,

$$\exp(\mathrm{ad}(tX))(tY) = \mathrm{Ad}(\exp(tX))(tY)$$

$$\| $$

$$e^{\mathrm{ad}(tX)}(tY) = \left(I + t\,\mathrm{ad}(X) + \frac{t^2\,\mathrm{ad}(X)^2}{2!} + \cdots\right)(tY)$$

$$= tY + t^2\,\mathrm{ad}(X)(Y) + t^3(\quad)\cdots$$

On the other hand

$$\exp\left(\mathrm{Ad}(\exp(tX))(tY)\right)$$
$$\parallel$$
$$\exp\left(D\,C_{\exp(tX)}(e)\,(tY)\right) = \cancel{\exp}\; C_{\exp(tX)}(\exp(tY))$$
$$= \exp(tX)\exp(tY)\exp(-tX)$$

By the exponential conj. formula

$$= \exp\left(tY + t^2[X,Y] + O(t^3)\right)$$

Hence, for t suff small, we conclude

$$\mathrm{Ad}(\exp(tX))(tY) = tY + t^2[X,Y] + O(t^3)$$
$$= tY + t^2\,\mathrm{ad}(X)(Y) + O(t^3)$$
$$\therefore\ \mathrm{ad}(X)(Y) = [X,Y] \quad \forall\, X, Y \in \mathfrak{g}. \ \boxtimes$$

<u>Def^n</u>: Let $\mathfrak{g}$ be a Lie algebra. An ideal of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{I} \subseteq \mathfrak{g}$ such that $\mathrm{ad}(X)(\mathfrak{I}) \subseteq \mathfrak{I}$ $\forall\, X \in \mathfrak{g}$.

<u>Example</u>:- $\left\{\begin{pmatrix}\alpha & & 0\\ & \ddots & \\ 0 & & \alpha\end{pmatrix} \middle| \alpha \in \mathbb{R}\right\}$ is an ideal of $\mathfrak{gl}_n$; $\mathfrak{sl}_n = \{X \mid \mathrm{tr}(X) = 0\} \subseteq \mathfrak{gl}_n$ is an ideal.

19-5 We saw that Lie subalgebras of $\mathfrak{g}$ correspond to connected Lie subgroups of G. What do ideals correspond to?

<u>Thm</u>:- Let H be a connected Lie subgroup of G and H be a normal subgroup. Then $\mathfrak{h} = \mathrm{Lie}(H)$ is an ideal in $\mathfrak{g} = \mathrm{Lie}\, G$.

*<u>Converse</u>: If G be <u>connected</u>, $\mathfrak{h} = \mathrm{Lie}\, H$ is ~~be~~ an ideal, then H is necessarily normal.

<u>Pf</u>:- Let $H \trianglelefteq G$. Let $Y \in \mathfrak{h}$ & $X \in \mathfrak{g}$. Then $\exp(tY) \in H \;\forall\, t \in \mathbb{R}$. Using normality, we have, for $s, t \in \mathbb{R}$

$$\exp(sX)\exp(tY)\exp(-sX) \in H$$

- For any $g \in G$, we consider the $\circlearrowleft$ diag.

$$\begin{tikzcd} G \arrow[r, "C_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}(g)"] & \mathfrak{g} \arrow[u, "\exp"] \end{tikzcd}$$

to get $g\exp(tY)g^{-1} = \exp(t\,\mathrm{Ad}(g)(Y))$.

Since $\exp$ is a local diffeo: $\mathfrak{g} \to G$, + we deduce that $\exp(t\,\mathrm{Ad}(g)(Y)) \in H$.

- The curve $\gamma(t) = \exp(t\,\mathrm{Ad}(g)(Y))$ is a 1-param subgroup of H corresp. to the left invariant vector field $\mathrm{Ad}(g)(Y)$ on H.

For $g = \exp(sX)$,

$$\mathrm{Ad}(\exp(sX))(Y) = e^{s\,\mathrm{ad}(X)}(Y) \in \mathfrak{h}$$

$\forall s \in \mathbb{R}$, by considering

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

Note now that the curve

$s \longmapsto e^{s\,\mathrm{ad}(X)}(Y)$ has its tangent $\mathrm{ad}(X)(Y)$ on $\mathfrak{h}$ at 0. Hence $[X,Y] \in \mathfrak{h}$.

*<u>Converse</u> : discussion hour.

- G be connected. Then $\ker(\mathrm{Ad}) = Z(G)$.
- $Z(G)$ is a closed Lie subgroup &
- $\mathrm{Lie}(Z(G)) = \{X \in \mathfrak{g} \mid \mathrm{ad}(X)(Y) = 0 \ \forall Y \in \mathfrak{g}\}$.

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* $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal. Then $\mathrm{ad}(\mathfrak{g})(\mathfrak{h}) \subseteq \mathfrak{h}$. Hence, for $\rho = \mathrm{Ad}$, $D\rho(e) = \mathrm{ad}$ satisfies $D\rho(e)(X)(\mathfrak{h}) \subseteq \mathfrak{h}$.

<u>Exercise</u>: 1. Let $\rho : G \to GL(V)$ be a repn of Lie groups; $W \subseteq V$ subspace s.t. $\rho(g)(W) \subseteq W \ \forall g \in G$. We call W a <u>G-inv. subspace</u>. Prove for a G-inv. subspace W, we have $D\rho(e)(X)(W) \subseteq W \ \forall X \in \mathfrak{g} = \mathrm{Lie}\, G$.

2. G be connected, $\rho : G \to GL(V)$ as above, $W \subseteq V$ a $\mathfrak{g}$-inv. subsp, i.e. $D\rho(e)(X)(W) \subseteq W \ \forall X \in \mathfrak{g}$. Then W is G-inv.

$\to$ Hence (2) $\Rightarrow \rho(g)\mathfrak{h} \subseteq \mathfrak{h}$ i.e. $\mathrm{Ad}(g)\mathfrak{h} \subseteq \mathfrak{h}$. This $\Rightarrow g\exp(\mathfrak{h})g^{-1} = c_g(\exp(\mathfrak{h})) = \exp(dc_g(e)(\mathfrak{h})) = \exp(\mathrm{Ad}(g)\mathfrak{h}) \subseteq \exp(\mathfrak{h})$; $\langle \exp(\mathfrak{h}) \rangle$ is conn $\Rightarrow H \trianglelefteq G$.