$$Df(a) = (a_{ij}) \quad \text{where } a_{ij} = \frac{\partial f_i}{\partial x_j}\Big|_a .$$

This matrix is called the Jacobian matrix of f at a.

Examples :- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a diff. func^n. The derivative of f at $p \in \mathbb{R}^n$, $Df(p)$ is a $1 \times n$ matrix, i.e. a row vector, called the gradient of f at p

$$(\nabla f)\big|_p = \left(\frac{\partial f}{\partial x_1}\Big|_p , \ldots , \frac{\partial f}{\partial x_n}\Big|_p \right).$$

The level sets of f are the sets $f^{-1}(c)$, $c \in \mathbb{R}$. For generic c, these give hyper-surfaces in $\mathbb{R}^n$, i.e. geom. objects having dim $n-1$, $\subseteq \mathbb{R}^n$. Since f is differentiable, for generic c, these objects are 'differentiable'.

To make sense of this, assume that $p \in S := f^{-1}(0)$ is such that $\rightarrow$

$\nabla f|_p \neq 0$. Then $\nabla f|_p$ would constitute a normal at p to the hyper-surface $S \subseteq \mathbb{R}^n$. If $\nabla f|_p \neq 0$, we can define

$$T_p S = \{(b_1, \dots, b_n) \in \mathbb{R}^n \mid \langle \nabla f|_p, (b_1 - a_1, \dots, b_n - a_n) \rangle = 0\}$$

where $p = (a_1, \dots, a_n)$.

$$= \left\{(b_1, \dots, b_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p \cdot (b_i - a_i) = 0\right\},$$

the tangent space to S at p.

<u>Remark</u>:- There are examples (?) of $f: \mathbb{R}^n \to \mathbb{R}^m$ such that $\frac{\partial f_i}{\partial x_j}\Big|_a$ exist $\forall\, i, j$, yet f is not continuous at a.

$\rightarrow$ $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff at $a \Rightarrow f$ is cont. at a.

<u>Thm</u>:- $Df(a)$ exists if all $\partial f_i / \partial x_j$ exist at all points in an open nbd of a <u>and</u> if each $\partial f_i / \partial x_j$ is continuous at a. $\boxtimes$