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Chain rule :- If $f: U \to \mathbb{R}^m$ and $g: V \to \mathbb{R}^l$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open and $f(U) \subseteq V$; f is diff at $a \in U$, g is diff at $f(a) \in V$, then the composite $g \circ f : U \to \mathbb{R}^l$ is diff. at a and its derivative at a is given by

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Df(a)"] \arrow[rr, bend right, "Dg(f(a)) \circ Df(a) = D(g \circ f)(a)"'] & \mathbb{R}^m \arrow[r, "Dg(f(a))"] & \mathbb{R}^l \end{tikzcd}$$

→ If $L: \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, i.e. $\exists$ $T: \mathbb{R}^n \to \mathbb{R}^m$ linear & $v \in \mathbb{R}^m$ with $L(x) = T(x) + v$ $\forall$ $x \in \mathbb{R}^n$; then L is differentiable and $DL(a) = T$ $\forall$ $a \in \mathbb{R}^n$.

→ $f, g : U \to \mathbb{R}^m$ be diff at $a \in \underset{\text{open}}{U} \subseteq \mathbb{R}^n$; $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is diff at a.

→ $f, g : U \to \mathbb{R}$ diff at $a \in U \Rightarrow f \cdot g$ diff at a. If $f(x) \neq 0$ $\forall$ $x \in U$, then $x \mapsto f(x)^{-1}$ is also diff at a.

→

→ Hence any polynomial function $P: \mathbb{R}^n \to \mathbb{R}$ is diffrentiable at all points; a rational function $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = P(x)/Q(x)$ where P & Q are polynomials, is differentiable on the open set $U = \mathbb{R}^n - Z(Q)$,
$Z(Q) = \{x \in \mathbb{R}^n \mid Q(x) = 0\}$.

<u>Exercise</u> :- 1. Identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Then $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is open. Prove that $i: GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$
$X \longmapsto X^{-1}$ is diffrentiable at all points in $GL_n(\mathbb{R})$.

2. Consider $f: \mathbb{R}^2 \to \mathbb{R}$; $(x,y) \longmapsto \begin{cases} \dfrac{x|y|}{(x^2+y^2)^{1/2}}, & (x,y) \neq (0,0) \\ 0, & x = y = 0. \end{cases}$

show that f is not diff at $(0,0)$, both partial derivatives exist at $(0,0)$.

• show that restriction of f to every line passing through $(0,0)$ is diff.

—— x —— x —— x —— x ——