Lecture-2

<u>A review of calculus</u>: Though the basic objects of study in this course are manifolds, these are locally Euclidean, i.e. every point has a neighbourhood $\simeq$ an open ball in $\mathbb{R}^n$. This allows us to do calculus on such abstract objects, using what are called chart maps and transition maps. We therefore recall first the <u>calculus on $\mathbb{R}^n$</u>: A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} =: \lambda$ exists; then we define the derivative f' of f at a to be λ, i.e. $f'(a) = \lambda$.

→ • The derivative of f at a represents the best linear approximation of f at a.

The above may be <u>rephrased</u> as: $f: \mathbb{R} \to \mathbb{R}$ is diff at a with derivative $\lambda = f'(a)$ if

$$\lim_{x \to a} \frac{f(x) - f(a) - \lambda \cdot (x - a)}{x - a} = 0.$$

→

• Since any $\mathbb{R}$-linear map $\phi : \mathbb{R} \to \mathbb{R}$ is of the form $\phi(x) = \alpha x$ for some $\alpha \in \mathbb{R}$ fixed, we may think of $\lambda \cdot (x-a)$ above as $\phi(x-a)$ for $\phi : \mathbb{R} \to \mathbb{R}$, $x \mapsto \lambda \cdot x$.

This leads & motivates us to define

<u>Def^n</u>: A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ if $\exists$ a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - \lambda(h) \|}{\|h\|} = 0$$

(Note that the norms on numerator & denom are in $\mathbb{R}^m$ & $\mathbb{R}^n$ resp.).

<u>Exercise</u>: 1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be diff at $a \in \mathbb{R}^n$ and $T_i : \mathbb{R}^n \to \mathbb{R}^m$ be linear maps, $i = 1, 2$, such that

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - T_i(h) \|}{\|h\|} = 0$$

for $i = 1, 2$; then $T_1 = T_2$.

2. For sufficiently small $x \in \mathbb{R}^n$ we have

$$f(a+x) = f(a) + \lambda(x) + \|x\| \, \epsilon(x),$$ where $\epsilon(x)$ satisfies $\lim_{x \to 0} \epsilon(x) = 0$.

Notation :- The linear map λ as above, if it exists, is denoted by $Df(a)$ and is called the derivative of f at a.

So $Df(a)$ is a linear transformation $: \mathbb{R}^n \to \mathbb{R}^m$.

Exercise : For any $x \in \mathbb{R}^n$,

$$Df(a)(x) = \lim_{t \to 0} \frac{1}{t}\left[f(a+tx)-f(a)\right].$$

Computing $Df(a)$:- For this, we'll compute the matrix of $Df(a) \in M_{m\times n}(\mathbb{R})$, where we write $Df(a) = [C_1 \cdots C_n]$, C_i are the columns of $Df(a)$, and $C_i := Df(a)(e_i)$.

By the above exercise

$$Df(a)(e_j) = \lim_{t \to 0} \frac{1}{t}\left[f(a+te_j) - f(a)\right]$$

→ We can write $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of its coordinate functions $f = (f_1, \cdots, f_m)^t$, $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le m$; so $f = \sum_{j=1}^{m} f_j e_j'$,

$e_j' = (0, \cdots, 0, \overset{j^{th}}{1}, 0, \cdots 0)^t$.

Hence we have $\rightsquigarrow$

$$f(a+te_j)-f(a)=\begin{pmatrix} f_1(a+te_j)-f_1(a)\\ \vdots \\ f_i(a+te_j)-f_i(a)\\ \vdots \\ f_m(a+te_j)-f_m(a)\end{pmatrix}$$

and $f_i(a+te_j)-f_i(a)$, for $a=\sum_{k=1}^{n} a_k e_k$

$$= f_i\left((a_1,\dots,a_{j-1},a_j+t,a_{j+1},\dots,a_n)^t\right)$$

$-f_i\left((a_1,\dots,a_j,\dots,a_n)^t\right)$. Thus, the

<u>i^{th} row</u> of the <u>j^{th} column</u> of $Df(a)$ is

$$\lim_{t\to 0}\frac{f_i((a+te_j))-f_i(a)}{t}=(ij)^{th}\text{ entry of}$$

the matrix of $Df(a)$.

<u>Def^n</u> (<u>Partial derivatives</u>): For $\varphi:\mathbb{R}^n\to\mathbb{R}$ and $a\in\mathbb{R}^n$, the partial derivative of f at $a=(a_1,\dots,a_n)^t$ with respect to x_i (the i^{th} coordinate) is the limit

$$\left.\frac{\partial\varphi}{\partial x_i}\right|_a=\lim_{t\to 0}\frac{\varphi((a_1,\dots,a_{i-1},a_i+t,a_{i+1},\dots,a_n)-\varphi((a_1,\dots,a_n))}{t},$$

if it exists. We therefore have $\rightsquigarrow$