# Lecture-20

**Multi-linear algebra / tensors** :- Discussion of higher order derivatives of smooth maps between manifolds naturally leads to the notion of tensors.

- For example if $U \subseteq \mathbb{R}^m$ be open & $\varphi: U \to \mathbb{R}^n$ a smooth map, we have, for $p \in U$, $D\varphi(p): T_pU \to T_{\varphi(p)}\mathbb{R}^n$ is an $\mathbb{R}$-linear map. Identifying $T_pU$ with $\mathbb{R}^m$, $T_{\varphi(p)}(\mathbb{R}^n) = \mathbb{R}^n$, we thus get

$$D\varphi: U \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n), \quad p \longmapsto D\varphi(p).$$

This is smooth, hence, for $p \in U$, $D^2\varphi(p): T_pU \to T_{\varphi(p)} \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n) \equiv M_{n \times m}(\mathbb{R})$ is $\mathbb{R}$-linear. So

$$D^2\varphi: U \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n))$$ and so on.

We'll study spaces like $\mathrm{Hom}_{\mathbb{R}}(V, \mathrm{Hom}_{\mathbb{R}}(V, W))$ and relate them to 'tensors'.

- **The dual**: For a (finite dim'l) vector space $V$ over a field $k$, the dual space $V^*$ is defined by $V^* = \mathrm{Hom}_k(V, k)$ = all $k$-linear maps: $V \to k$. Elements of $V^*$ are called linear forms/functionals. Given a basis of $V$, $\dim_k V = n < \infty$, say $\{e_1, \dots, e_n\}$, the functionals $\{e_1^*, \dots, e_n^*\}$

form the dual basis, $e_i^*(e_j) := \delta_{ij}$, $f = \sum_i f(e_i) e_i^*$ for any $f \in V^*$. So $\dim V^* = \dim V = \dim V^{**}$; $V \simeq V^{**}$ via $v \mapsto (f \mapsto f(v))$.

**Multi-linear maps**:- $V$ be a vector sp $/k$. An r-linear ~~map~~ form on $V$ is a map $\varphi : V \times \cdots \times V \to k$, linear in each variable.

**Example**:- Let $f_1, \cdots, f_r \in V^*$ and define

$$\varphi : V \times \cdots \times V \to k \text{ by } \varphi(v_1, \cdots, v_r) = \prod_{i=1}^{r} f_i(v_i).$$

Then $\varphi$ is r-linear. We denote this r-linear form on $V$ by $\varphi = f_1 \otimes f_2 \otimes \cdots \otimes f_r$ ($f_1$ tensor $f_2 \cdots$)

→ Let $L^r(V, k)$ = vector space of all r-linear forms on $V$. We have

**Lemma**:- For $\varphi \in L^r(V,k)$, $\varphi = \sum_{1 \le i_1, \cdots, i_r \le n} \varphi(e_{i_1}, \cdots, e_{i_r}) e_{i_1}^* \otimes \cdots \otimes e_{i_r}^*$.

$\{e_{i_1}^* \otimes \cdots \otimes e_{i_r}^* ; 1 \le i_1, \cdots, i_r \le n\}$ is a basis for $L^r(V,k)$.

**Pf**: Exercise!

- $\operatorname{Dim}_k L^r(V,k) = n^r$. The space $L^r(V,k)$ is called the r-fold tensor product of $V^*$, denoted by $V^{*\otimes r} := \underbrace{V^* \otimes \cdots \otimes V^*}_{r} = T^r(V^*)$.

More generally, for $V_1, \cdots, V_r$ vector spaces, we define $V_1^* \otimes \cdots \otimes V_r^*$ as the space of $r$-linear forms $V_1 \times \cdots \times V_r \to k$

$$\mathcal{L}(V_1, \ldots, V_r; k) \equiv V_1^* \otimes \cdots \otimes V_r^*.$$

The forms $\left\{ e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes \cdots \otimes e_{i_r}^{r*} \;\middle|\; \begin{array}{l} 1 \leq i_1 \leq \dim V_1 \\ 1 \leq i_2 \leq \dim V_2 \\ \vdots \\ 1 \leq i_r \leq \dim V_r \end{array} \right\}$

is a basis of $V_1^* \otimes \cdots \otimes V_r^*$.

Pf: Exercise.

For $V_1, \ldots, V_r$ finite dim'l $k$-vect spaces, we define $V_1 \otimes \cdots \otimes V_r = (V_1^*)^* \otimes \cdots \otimes (V_r^*)^*$

& $v_1 \otimes \cdots \otimes v_r$ is given by

$$(v_1 \otimes \cdots \otimes v_r)(f_1, \ldots, f_r) = \prod_{i=1}^{r} f_i(v_i).$$

- Given $T \in \mathrm{Hom}_k(V, W)$ $\exists!$ $T^{\otimes r} \in \mathrm{Hom}(V^{\otimes r}, W^{\otimes r})$ such that $T^{\otimes r}(v_1 \otimes \cdots \otimes v_r) = T(v_1) \otimes \cdots \otimes T(v_r)$
- In $V \otimes W$, we have

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$\alpha v \otimes w = v \otimes \alpha w = \alpha (v \otimes w), \alpha \in k$$
$$v \otimes (w \otimes z) = (v \otimes w) \otimes z \quad \text{in } V \otimes W \otimes Z.$$

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Alternating forms: Let char$(k) \neq 2$, $V$ a $k$-vector space. An $r$-linear form $\varphi: \underbrace{V \times \cdots \times V}_{r} \to k$ s.t.

$$\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r)$$

$$= -\varphi(v_1, \ldots, v_{i-1}, v_j, \ldots, v_{j-1}, v_i, \ldots, v_r)$$

(obtained by flipping $i$ & $j$).

Let $\Lambda^r(V) = \{ \varphi \in T^{\otimes r}(V) \mid \varphi: V^* \times \cdots \times V^* \to k$ is alternating$\}$.

$\subseteq T^{\otimes r}(V)$.

We have a natural map $T^r(V) \xrightarrow{\text{Alt}} \Lambda^r(V)$ obtained by 'averaging'

$$\psi \longmapsto \left( (v_1, \ldots, v_r) \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} \text{Sign}(\sigma) \cdot \psi(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \right)$$

Then

- Alt is $k$-linear, if $\psi \in \Lambda^r(V)$ then

$\text{Alt}(\psi) = \psi$, for $\psi \in T^r(V)$, $\text{Alt}(\text{Alt}(\psi)) = \text{Alt}(\psi)$.

→ Wedge product :- We have a normalized product $\Lambda^r(V) \times \Lambda^s(V) \to \Lambda^{r+s}(V)$

$$(\varphi, \psi) \longmapsto c \cdot \text{Alt}(\varphi \otimes \psi)$$

where $c = \dfrac{(r+s)!}{r!\, s!}$, denoted by $\varphi \wedge \psi$.

- $V \otimes W$ is naturally isom to $\mathrm{Hom}_k(V^*, W)$.

The map $V \otimes W \to \mathrm{Hom}_k(V^*, W)$ such that

$$v \otimes w \longmapsto (f \mapsto f(v)w)$$ gives an

isom. Here $V, W$ are both assumed finite dim'l.

<u>Cor!</u> $V^* \otimes V \simeq \mathrm{Hom}_k(V, V) = \mathrm{End}_k(V)$.

<u>Symmetric/skew-symmetric $\equiv$ alternating tensors:</u>

Let $t \in V^{\otimes n}$ & $\sigma \in S_n$. We define

$\sigma(t)$ by $\sigma(t)(f_1, \dots, f_n) = t(f_{\sigma(1)}, \dots, f_{\sigma(n)})$,

$f_i \in V^*$. We call $t$ symmetric if $\sigma(t) = t$

$\forall \sigma \in S_n$. Let $\mathrm{Sym}^n(V)$ = all symmetric tensors.

<u>Symmetrization map</u>: $V^{\otimes n} \longrightarrow \mathrm{Sym}^n(V)$

$t \mapsto \mathrm{Sym}^n(t)$, where

$$\mathrm{Sym}^n(t)(f_1, \dots, f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} t(f_{\sigma(1)}, \dots, f_{\sigma(n)}).$$

<u>Properties of $\wedge$</u> :- • $(\alpha \varphi_1 + \beta \varphi_2) \wedge \psi = \alpha(\varphi_1 \wedge \psi) + \beta(\varphi_2 \wedge \psi)$

- $\varphi \wedge (\alpha \psi_1 + \beta \psi_2) = \alpha\, \varphi \wedge \psi_1 + \beta\, \varphi \wedge \psi_2$
- $\underset{\Lambda^k V}{\varphi} \wedge (\underset{\Lambda^l V}{\psi} \wedge \underset{\Lambda^r V}{\eta}) = (\varphi \wedge \psi) \wedge \eta = \frac{(k+l+r)!}{k!\, l!\, r!} \mathrm{Alt}(\varphi \otimes \psi \otimes \eta)$ ; $\varphi \wedge \psi = (-1)^{kl} \psi \wedge \varphi$.

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**Thm**: Let $\{e_1, \cdots, e_n\}$ be a basis of $V$. Then $\{e_{i_1} \wedge \cdots \wedge e_{i_r}, 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$ is a basis of $\Lambda^r V$. Hence $\dim_k \Lambda^r V = {}^nC_r$.

**Cor**: $\dim \Lambda^n V = 1$, $n = \dim V$.

**Prf**: Let $\varphi \in \Lambda^r(V)$. Then $\varphi \in \bigotimes^r V$, so

$$\varphi = \sum_{i_1, \cdots, i_r} a_{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}.$$ Since $\varphi \in \Lambda^r V$,

$$\varphi = \mathrm{Alt}(\varphi) = \sum a_{i_1 i_2 \cdots i_r} \mathrm{Alt}(e_{i_1} \otimes \cdots \otimes e_{i_r})$$

$$= \sum a'_{i_1 i_2 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}, \quad a'_* = \text{mult of } a_*.$$

If $\sum \beta_{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} = 0$, then, for a fixed $(j_1, \cdots, j_r)$, $\sum \beta_{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} (e^*_{j_1}, \cdots, e^*_{j_r}) = 0$

$$\Downarrow$$

$$\sum \beta_{i_1 \cdots i_r} e_{i_1}(e^*_{j_1}) \cdots e_{i_r}(e^*_{j_r}) = 0$$

$\therefore \beta_{j_1 \cdots j_r} = 0$ etc. $\boxtimes$

- Let $T \in \mathrm{End}(V)$. Then $\exists!$ linear map $\Lambda^r T : \Lambda^r V \to \Lambda^r V$ such that $\Lambda^r(T)(v_1 \wedge \cdots \wedge v_r) = Tv_1 \wedge \cdots \wedge Tv_r$.

<u>Cor</u>: Since $\dim \Lambda^n V = 1$, $n = \dim V$, for $T \in \mathrm{End}(V)$, $\Lambda^n T : \Lambda^n V \to \Lambda^n V$ must be a scalar mult. In fact $\Lambda^n T = \det(T)$.

<u>Pf</u>: Let $\{e_1, \dots, e_n\}$ be a basis of $V$ and

$$Te_j = \sum_j a_{ij} e_j \ . \quad \text{Then } \Lambda^n T(e_1 \wedge \dots \wedge e_n)$$

$$= Te_1 \wedge \dots \wedge Te_n = \left(\sum_i a_{i_1} e_i\right) \wedge \left(\sum_i a_{i_2} e_i\right) \dots \wedge \sum_i a_{i_n} e_i\Big)$$

$$= \sum_{j_i \neq j_k \ i \neq k} a_{j_1 1} a_{j_2 2} \dots a_{j_n n} \, e_{j_1} \wedge \dots \wedge e_{j_n}$$

$$= \sum a_{j_1 1} \dots a_{j_n n} \, \mathrm{sign}(j_1, \dots, j_n) \, e_1 \wedge \dots \wedge e_n$$

$$= \det(T) \, e_1 \wedge \dots \wedge e_n . \qquad \boxtimes$$