Lecture-21

Differential forms on manifolds :- These are certain objects associated to a manifold for which we can define a notion of integral, use them to define orientations, among other applications.

Let M be a manifold. Then $C^\infty(M)$ is a commutative $\mathbb{R}$-algebra and $\mathfrak{X}(M)$ is a $C^\infty(M)$-module via $(f \cdot X)(p) = f(p)X(p)$.

We denote $\mathfrak{X}(M)$ by $\mathcal{D}_1(M)$ when treated as a $C^\infty(M)$-module. Let $\mathcal{D}^1(M)$ denote the $C^\infty(M)$ dual of $\mathcal{D}_1(M)$, i.e.

$$\mathcal{D}^1(M) := \operatorname{Hom}_{C^\infty(M)}(\mathcal{D}_1(M), C^\infty(M)),$$

the $C^\infty(M)$-module of $C^\infty(M)$-linear forms on $\mathcal{D}_1(M)$. We call elements of $\mathcal{D}^1(M)$ as <u>differential</u> <u>1-forms</u> on M.

<u>Example</u> :- Let $f \in C^\infty(M)$. Define

$$df : \mathcal{D}_1(M) \to C^\infty(M) \quad \text{by } df(X) = X(f).$$

Then $df \in \mathrm{Hom}_{C^\infty(M)}(D_1(M), C^\infty(M)) =: D^1(M)$, and $\omega = df$ is a differential 1-form. We'll discuss this further (we have done this before!)

<u>Recall</u> the co-tangent space of M is $T_p(M)^* =$ vector space dual of T_pM. For $f \in C_p^\infty(M)$, we can define $df(p) : T_pM \to \mathbb{R}$ by

$$df(p)(v) := v(f). \text{ So } df(p) \in T_pM^*.$$

Let $\{x_i\}$ be local coordinates around p. Then $x_i : U \to \mathbb{R}$ are smooth and $\{dx_i(p)\}$ is a basis of T_pM^*, this is the dual basis corresponding to the basis $\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}$ of T_pM. For $v \in T_pM$ & $f \in \cancel{C_p^\infty}\, C^\infty(M)$, $v = \sum v(x_i)\frac{\partial}{\partial x_i}\Big|_p$. Hence

$$df(p)(v) = v(f) = \sum v(x_i)\frac{\partial f}{\partial x_i}\Big|_p$$

$$= \sum \frac{\partial f}{\partial x_i}\Big|_p v(x_i)$$

$$= \sum \frac{\partial f}{\partial x_i}\Big|_p dx_i\Big|_p (v). \text{ Hence}$$

$$df(p) = \sum \frac{\partial f}{\partial x_i}\Big|_p dx_i\Big|_p.$$

21-3

For $\omega \in \mathcal{D}^1(M) = \mathrm{Hom}_{C^\infty(M)}(\mathcal{D}_1(M), C^\infty(M))$, we wish to get a local expression. We proceed to do this now. For $p \in M$, define $\omega_p \in T_pM^*$ by $\omega_p(v) = \omega(X)(p)$ where $v \in T_pM$ and X is a smooth vector field on M with $X(p) = v$.

<u>ω_p is well defined</u> :- We have seen before that if $X = 0$ on an open set $V \subseteq M$, then $\omega(X) = 0$ on V. For this, take $p \in V$ and f a smooth function ~~in an nbd of p~~ s.t. $f = 0$ ~~on that~~ in a nbd of $p \subseteq V$, $f = 1$ outside V. Then $\boxed{fX = X}$. Since ω is $C^\infty(M)$-linear, we have $\omega(fX) = f\,\omega(X)$. So $\omega(X)(p)$
$= \omega(fX)(p) = f(p)\,\omega(X)(p) = 0$.

Hence $X = 0$ on $V \Rightarrow \omega(X) = 0$ on V.

Thus ω induces a 1-form on an open subset U of M. If $X = \sum_i X(x_i) \frac{\partial}{\partial x_i}$ on U, then $X(p) = 0 \Leftrightarrow X(x_i)(p) = 0 \;\forall i$.

Hence

$$\omega(X)(p) = \left[\sum_i X(x_i)\,\omega\left(\frac{\partial}{\partial x_i}\right)\right](p)$$

$$= \sum_i X(x_i)\,\underbrace{\omega\left(\frac{\partial}{\partial x_i}\right)(p)}_{} = 0$$

Hence if $X(p) = 0$ for some $p \in M$, then $\omega(X(p)) := \omega(X)(p) = 0$.

Hence ω_p defines as above makes sense.

→ Let $D^1(p) := \{\omega_p \mid \omega \in D^1(M)\}$.

→ $D^1(p) = T_pM^*$. Clear: $D^1(p) \subseteq T_pM^*$.

Conversely, for $\varphi \in T_pM^*$, set $\varphi_i = \varphi\left(\frac{\partial}{\partial x_i}\Big|_p\right)$, $\Psi_U := \sum_i \varphi_i \, dx_i$ on $(U, x) \ni p$. Then

→ $\exists\ \Psi \in D^1(M)$ s.t. $\Psi_U = \Psi$ on an open $V \subseteq U$, $p \in V$.

For this choose $V \ni p$, $\overline{V}$ compact $\subseteq U$; f smooth on M s.t. f has compact support $\subseteq U$, $f = 1$ on V. Define Ψ by $\Psi(X) := f\Psi_U(X)$

21-5

on U & $\psi(x)=0$ on $M \setminus U$. etc.

$\rightarrow \quad \psi_p = \varphi$.

Hence $\mathcal{D}^1(M)$ consists of elements ω which can be thought of as a map $p \mapsto \omega_p \in T_pM^*$

, $\omega : M \to \bigcup_p T_pM^*$, $p \mapsto \omega_p \in T_pM^*$.

We'll define higher diff forms next.
