Lecture-22

In the last lecture we discussed the space of all differential 1-forms on M, as the set $D^1(M)$ consisting of elements ω, thought of as a map $p \longmapsto \omega_p$, $\omega : M \to \bigcup_p T_pM^*$ ← cotangent bundle of M, $\omega_p \in T_pM^*$.

Higher degree differential forms :- Recall the $C^\infty(M)$-module $D_1(M) = \mathcal{X}(M)$. We define a differential form of degree r on M to be an r-linear alternating map of $C^\infty(M)$-modules $\omega : D_1(M) \times \cdots \times D_1(M) \longrightarrow C^\infty(M)$. We denote the set of all deg r diff forms on M by $D^r(M)$.

Example :- Let $\omega \in D^1(M)$. Define a 2-form $d\omega : D_1(M) \times D_1(M) \to C^\infty(M)$ by

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Then $d\omega \in D^2(M)$.

Exercise : We defined, for $f \in C^\infty(M)$ the 1-form $df \in D^1(M)$. What is $d(df)$?

This example is a particular case of

The d-operator: We'll define $d : D^r(M) \to D^{r+1}(M)$

For this, we use (any) local expression for ω. Let (U, x) be a coordinate chart on M. Then ω has an expression

$$\omega = \sum a_{i_1 \cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r} \quad \text{on } U.$$

($\omega_p \in \Lambda^r T_p M$, $\omega_p(v_1, \ldots, v_r) = \omega(X_1, \ldots, X_r)(p)$, $X_i(p) = v_i$ on U etc)

Define

$$d^L\omega := \sum da_{i_1 \cdots i_r} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}$$

(here L stands for 'local').

We would like to extend the def^n of d^L to all of M. So we must verify, for another chart (V, y) on M with $U \cap V \neq \phi$, if

$$\omega = \sum b_{j_1 \cdots j_r} dy_{j_1} \wedge \cdots \wedge dy_{j_r} \quad \text{on } V, \text{ then}$$

$$\sum db_{j_1 \cdots j_r} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_r} = \sum da_{i_1 \cdots i_r} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}$$

on $U \cap V$.

Global definition of d: We'll define $d : \bigoplus_{r=0}^{m} D^r M \to \bigoplus D^r M$ and show it matches with d^L locally.

22-3

We have $D^0 M = C^\infty(M)$. Let $\omega \in D^r(M)$, $r \geq 1$ and $X_1, \dots, X_{r+1} \in D_1(M)$. Define

$$d\omega(X_1, \dots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} X_i\, \omega(X_1, \dots, \hat{X}_i, \dots, X_{r+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{r+1})$$

→ $d\omega$ is additive clearly.

- $C^\infty(M)$-linearity: $d\omega(X_1, \dots, fX_k, \dots, X_{r+1}) \overset{?}{=} f\, d\omega(X_1, X_2, \dots, X_k, \dots, X_{r+1})$.

- So $d\omega$ is $C^\infty(M)$-linear
- $d\omega$ is alternating, so $d\omega \in D^{r+1}(M)$.

$*_1$: $\forall f \in C^\infty(M)$, $df(X) = X(f)$.

$*_2$: d is $\mathbb{R}$-linear, $d(\alpha\omega + \beta\eta) = \alpha d\omega + \beta d\eta$

$*_3$: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$ for $\omega \in D^r(M)$, $\eta \in D^s(M)$.

$*_4$: $d^2 = 0$, i.e. $d(d\omega) = 0 \; \forall \omega$.

Proposition: Any $C^\infty(M)$-linear operator $d' : D^r(M) \to D^{r+1}(M)$ satisfying the properties $*_1 - *_4$ must coincide with d^L locally.

Proof:- We have

$$d'(dx_{i_1}\wedge\cdots\wedge dx_{i_r}) = d'(dx_{i_1})\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_r}$$
$$- dx_{i_1}\wedge d'(dx_{i_2})\wedge dx_{i_3}\wedge\cdots\wedge dx_{i_r}$$
$$+ \cdots$$

$$d'f = df \quad \forall f \in C^\infty(M) \qquad (*_1)$$

$\Rightarrow \; d'(dx_{i_k}) = d'^2(x_{i_k}) = 0 \;\forall k \; (*_4)$. Hence

$$d'(f\,dx_{i_1}\wedge\cdots\wedge dx_{i_r}) = df\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_r}$$
$$= d^L(f\,dx_{i_1}\wedge\cdots\wedge dx_{i_r}).$$

$\boxtimes$

Hence, to check the listed properties, it is necessary that d^L satisfies the properties on any chart (U,x). We have, for $f \in C^\infty(M)$, $d^L(f) = df$ (by def^n of d^L), and, we have seen that for any $X \in \mathfrak{X}(M) = D_1(M)$, $df(X) = X(f)$. So $*_1$ is satisfied by d^L; $*_2$ is obvious. For $*_3$, using linearity, it suffices to check $*_3$ for $\omega = a\,dx_{i_1}\wedge\cdots\wedge dx_{i_r}$ & $\eta = b\,dx_{j_1}\wedge\cdots\wedge dx_{j_s}$. Then

$$\omega\wedge\eta = a\cdot b\,dx_{i_1}\wedge\cdots\wedge dx_{i_r}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_s}$$

Hence $d^L(\omega\wedge\eta) = d(ab)\wedge dx_{i_1}\wedge\cdots\wedge dx_{j_s}$

$$=(da\cdot b + a\cdot db)\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_r}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_s}$$

$$= da\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_r}\wedge b\,dx_{j_1}\wedge\cdots\wedge dx_{j_s}$$

$$+ a\,dx_{i_1}\wedge\cdots\wedge dx_{i_r}\wedge(-1)^r db\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_s}$$

$$= d\omega\wedge\eta + (-1)^r\omega\wedge d\eta.$$ Hence $*_3$ holds for d^L.

$*_4$: Exercise: Using $\frac{\partial^2}{\partial x_i\partial x_j} = \frac{\partial^2}{\partial x_j\partial x_i}$ and $dx_i\wedge dx_j = -dx_j\wedge dx_i$, it follows that $d_L^2(f) = 0 \quad \forall f\in C^\infty(M)$. ☒

Since ω in general is a sum of (wedge) products of r 1-forms of type df, using $*_3$, every term of $d_L^2(\omega)$ has a factor of the form $d^{L^2}(f)=0$, hence it follows that $d_L^{L^2}(\omega)=0$. (Check details) ☒

Theorem: $d: D^r(M)\to D^{r+1}(M)$ defined by

$$d\omega(X_1,\ldots,X_{r+1}) = \sum_{i=1}^{r+1}(-1)^{i+1}\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{r+1})$$

$$+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{r+1})$$

22-6

d matches with d^L on local charts.

Sketch :- Using linearity of d & d^L, it reduces to check, for $\omega = a\, dx_1 \wedge \cdots \wedge dx_r$,

$$d\omega\Big(\underset{\overset{\shortparallel}{X_{i_1}}}{\frac{\partial}{\partial x_{i_1}}}, \ldots, \underset{\overset{\shortparallel}{X_{i_r}}}{\frac{\partial}{\partial x_{i_r}}}\Big) = d^L\omega\Big(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_r}}\Big)$$

This follows by explicit computations. $\boxtimes$

Pull back of a differential form :- Let $\varphi : M \to N$ be a smooth map, δ a diff. form on N. We define the pull back $\varphi^*(\delta)$ to be the differential form on M, having same degree as δ (say s), by

$$\varphi^*(\delta)(X_1, \ldots, X_s)(p) = \delta_{\varphi(p)}\big(D\varphi(p)(X_1(p)), \ldots, D\varphi(p)(X_s(p))\big).$$

Then $\varphi^*(\delta) \in D^s(M)$. So $\varphi^* : D^s(N) \to D^s(M)$ $s \geq 0$. We have

Propⁿ :- $\varphi : M \to N$ smooth & $\omega \in D(N)$. Then $d(\varphi^*\omega) = \varphi^*(d\omega)$

22-7.

i.e. exterior derivative operator d commutes with pull-backs.

Pf: We may check this locally. So let

$\omega = a\, dy_1 \wedge \cdots \wedge dy_r$. Then $d\omega = da \wedge dy_1 \wedge \cdots \wedge dy_r$

and $\varphi^*(d\omega) = \varphi^*(da \wedge dy_1 \wedge \cdots \wedge dy_r)$

$$= \varphi^* da \wedge \varphi^* dy_1 \wedge \cdots \wedge \varphi^* dy_r$$

Writing $\varphi_i = y_i \circ \varphi$

$$= d(a \circ \varphi) \wedge d\varphi_1 \wedge \cdots \wedge d\varphi_r.$$

Also $\varphi^*\omega = \varphi^*(a\, dy_1 \wedge \cdots \wedge dy_r)$

$$= \varphi^*(a)\, \varphi^* dy_1 \wedge \cdots \wedge \varphi^* dy_r$$

$$= a \circ \varphi \cdot d\varphi_1 \wedge \cdots \wedge d\varphi_r.$$ Thus

$$d(\varphi^*\omega) = d(a \circ \varphi) \wedge d\varphi_1 \wedge \cdots \wedge d\varphi_r.$$

Hence φ^* and d commute. $\boxtimes$
