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So far, we have defined differential forms of degree r, $r \geq 0$; their space is denoted by $D^r(M)$. We have defined a d-operator : $D^r(M) \xrightarrow{d_r} D^{r+1}(M)$ having the property $d^2 = 0$, i.e. $d_{r+1} \circ d_r = 0$, $r \geq 0$. Note also that d is an $\mathbb{R}$-linear map.

Cochain complexes & their cohomology groups:

Let $\mathcal{C} : 0 \to C^0 \to C^1 \to \cdots \to C^{i-1} \xrightarrow{d_{i-1}} C^i \xrightarrow{d_i} C^{i+1} \to \cdots$

be a sequence of abelian groups and homomorphisms with $d_i \circ d_{i-1} = 0$ $\forall i$, i.e. $\operatorname{Image} d_{i-1} \subseteq \operatorname{Ker} d_i$. The i^{th} cohomology group of $\mathcal{C}$ is defined by $H^i(\mathcal{C}) = \dfrac{\operatorname{Ker} d_i}{\operatorname{Im}(d_{i-1})}$.

Elements of $\operatorname{Ker} d_i$ are called i-cocycles and elements of $\operatorname{Im}(d_{i-1})$ are called i-coboundaries.

These cohomology groups measure the failure of $\mathcal{C}$ to be exact at C^i; $H^i(\mathcal{C}) = 0 \iff \mathcal{C}$ is exact at C^i.

There is an analogous 'dual' notion of a chain complex: a sequence

$$\mathcal{C}: \cdots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

of abelian groups and homomorphisms, with $d_i \circ d_{i+1} = 0$ & decreasing degrees. One defines the i^{th} homology group $H_i(\mathcal{C}) = \dfrac{\operatorname{Ker}(d_i)}{\operatorname{Im}(d_{i+1})}$. This group measures the failure of $\mathcal{C}$ from being exact (i.e. $\operatorname{Ker} d_i = \operatorname{Im} d_{i+1}$) at C_i.

de-Rham cohomology groups: We use the co-chain complex

$$\cdots \to D^i(M) \xrightarrow{d_i} D^{i+1}(M) \to \cdots$$

of the space of differential forms. A diff form ω of degree k is called closed if $d\omega = 0$ and call ω exact if $\omega = d\eta$ for some $\eta \in D^{k-1}(M)$. Since $d_i \circ d_{i-1} = 0$, it follows that exact forms are closed. Let

$$H^k_{dR}(M) = \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}.$$

This is the de-Rham cohomology of M in degree k.

We may sketch a proof of

<u>Theorem</u> :- (i) $H^k_{dR}(M)$ are finite dim'l $\mathbb{R}$-vector spaces.

(ii) If $\beta_k = \dim_{\mathbb{R}} H^k_{dR}(M)$, then

$$\sum_k (-1)^k \beta_k = \chi(M), \text{ the Euler-}$$

characteristic of M; the number (integer) β_k is called the k^{th} Betti no. of M. $\boxtimes$

<u>Orientation</u> :- Recall, while handling 'limits of integration' we needed orientation. To do this for level surfaces $S = f^{-1}(c)$, we defined S to be <u>orientable</u> if the map $S \longrightarrow \mathbb{R}^{n+1}$ $p \longmapsto N(p) := \nabla f(p)$ is smooth; where we assume $\nabla f(p) \neq 0 \ \forall p \in S$. In other words, the normal varies smoothly as we move on S.

<u>Example</u>: $f : \mathbb{R}^3 \to \mathbb{R}$, $c = 0$, $\nabla f(p) \neq 0 \ \forall p \in S = f^{-1}(0)$. Then, for $p \in S$, $\nabla f(p) \perp T_p(S)$. Now $T_p(S)$ is 2 dim'l, say with basis $\{v_1, v_2\}$. Thus $\nabla f(p) \in \langle v_1 \wedge v_2 \rangle$ | Excercise | . $\rightarrow$

~~Moreover $p \mapsto \nabla f(p)$ is smooth.~~

- Orientable manifold: A smooth manifold M is defined to be orientable if $\exists$ a smooth differential $m = \dim M$-form ω on M such that $\omega_p \neq 0 \ \forall p \in M$.

We call M oriented if M is orientable and a top degree nowhere vanishing diff. form ω is fixed, called the orientation class of M.

→ A basis $\{v_i, 1 \leq i \leq m\}$ of T_pM is positively oriented if $\omega_p(v_1, \dots, v_m) > 0$.

- Orientability and coordinate systems: Let (M, ω) be an oriented manifold, $p \in M$ and let (U, x) be a coordinate chart around p $(x(p) = 0)$. For any $q \in U$, ω_q defines an orientation on T_qM and we can define positively or negatively oriented coordinate systems.

Call (U, x) positively oriented if $\omega_q\left(\frac{\partial}{\partial x_1}\Big|_q, \dots, \frac{\partial}{\partial x_m}\Big|_q\right) > 0 \ \forall q \in U$.

→ If $(x_1, \dots, x_m)$ is positively oriented then $(x_{\sigma(1)}, \dots, x_{\sigma(m)})$ is negatively oriented for any odd $\sigma \in S_m$.

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Exercise :- If U is connected, then $q \mapsto \omega_q\left(\frac{\partial}{\partial x_1}\Big|_q, \dots, \frac{\partial}{\partial x_m}\Big|_q\right)$ has the same sign on U and we can write

$$\omega = f_U \, dx_1 \wedge \dots \wedge dx_m \text{ on } U \text{ with}$$

$f_U(q) > 0 \ \forall q$ or $f_U(q) < 0 \ \forall q \in U$.

Remark :- If (U, x), (V, y) are two +vely (resp. -vely) oriented charts on M, then

$$\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0 \quad \text{on } U \cap V.$$

→ We call two charts (U, x), (V, y) consistently oriented if either $U \cap V = \phi$ or $U \cap V \neq \varphi$ and $\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$ on $U \cap V$.

→ We call an atlas $\mathcal{A}$ an oriented atlas if all its charts are consistently oriented.

Thm : M is orientable $\Leftrightarrow$ M admits an oriented atlas. $\boxtimes$