Lecture-24

Integration :- We'll discuss now a theory of integration for differential forms on manifolds.

Let $U, V \subseteq \mathbb{R}^n$ be open, $\varphi : U \xrightarrow{\simeq} V$ a C^1-diffeomorphism. For any $f : V \to \mathbb{R}$ continuous such that $\operatorname{Supp}(f) := \{x \in V \mid f(x) \neq 0\}$ is compact, we have

$$\int_V f(y)\,dy = \int_U (f\circ\varphi)\,|\det D\varphi(x)|\,dx$$

More generally: for $U \subseteq \mathbb{R}^n$ open, $\varphi : U \to \mathbb{R}^n$ a 1-1 C^1 map with $\det(D\varphi(x)) \neq 0$ $\forall x \in U$, if $f : \varphi(U) \to \mathbb{R}$ is integrable, then

- $$\int_{\varphi(U)} f(y)\,dy = \int_U (f\circ\varphi)\,|\det(D\varphi(x))|\,dx.$$

This is called the change of variable formula.

- Exercise: Assume U, hence $V := \varphi(U)$ are

Connected. ~~(φ is assumed smooth~~ Then φ is <u>orientation</u> <u>preserving</u> or <u>orientation reversing</u>

i.e. $\det(D\varphi(x)) > 0 \ \forall x$ or $\det(D\varphi(x)) < 0$

$\forall \ x \in U$.

Hence, in the case U is connected,

$$\int_{V=\varphi(U)} f(y)\,dy = \pm \int_U (f\circ\varphi)\det(D\varphi(x))\,dx.$$

Next, <u>suppose f is smooth</u> and consider the differential form $\omega := f(y)\,dy_1\wedge\cdots\wedge dy_n$ on V, y_i the usual coordinates on $\mathbb{R}^n|_V$.

The pull back $\varphi^*(\omega)$ is then

$$\varphi^*(\omega) = (f\circ\varphi)\,\varphi^*(dy_1\wedge\cdots\wedge dy_n)$$

$$\cancel{= (f\circ\varphi)\, d(x_1\circ\varphi}$$

Now $\varphi^* : \Lambda^n(T_{\varphi(p)}V)^* \longrightarrow \Lambda^n(T_pU)^*$ is an isomorphism. Hence the basis elements $\varphi^*(dy_1\wedge\cdots\wedge dy_n)$ and $dx_1\wedge\cdots\wedge dx_n$ are related as

$$\varphi^*(dy_1\wedge\cdots\wedge dy_n) = \det D\varphi(x)\cdot dx_1\wedge\cdots\wedge dx_n$$

So $\varphi^*(\omega) = (f\circ\varphi)\det(D\varphi(x))\,dx_1\wedge\cdots\wedge dx_n$

We can thus interpret / rewrite the change of variable formula as $(\dot{\circ})$

$$\int_{V=\varphi(U)} \omega = \pm \int_U \varphi^*(\omega).$$

On M, we can define the integral of a smooth differential form ω of degree $m = \dim M$, so long as $\underline{\text{support}\,(\omega)}$

$$:= \{p\in M \mid \omega_p \neq 0\},$$ is compact,

$\text{supp}\,(\omega) \subseteq (U, x)$, the local chart being (U, φ^{-1}); we can write $\omega = f\,dx_1\wedge\cdots\wedge dx_m$ and define

$$\int_M \omega := \int_{\varphi^{-1}(U)} f(\varphi(p))\,dx_1\cdots dx_m \; ; \quad (1)$$

The RHS = Riemann int = Lebesgue int of $f\circ\varphi$. Since $\text{Supp}(f\circ\varphi) \subseteq \varphi(U)$, this is finite.

This is well defined : If (V, ψ^{-1}) is another chart with $\mathrm{Supp}(\omega) \subseteq V$, then

$$\int_M \omega = \int_{\psi^{-1}(V)} f(\psi(p))\, dy_1 \cdots dy_m \quad , \; y_i \text{ coords on } V.$$

②

If (V, ψ^{-1}) & (U, φ^{-1}) are *consistently oriented, then the RHS of ① & ② are equal, using (o). (i.e. $\det\left(\frac{\partial x_i}{\partial y_j}\right) > 0$).

In this case $\psi^{-1} \circ \varphi : \varphi^{-1}(U \cap V) \xrightarrow{\sim} \psi^{-1}(U \cap V)$ is orient^n preserving &

$$\int_{\psi^{-1}(U\cap V)} \psi^* \omega = \int_{\varphi^{-1}(U \cap V)} \varphi^* \omega$$

this is

→ Therefore to define integral of a top degree form having compact support, we need M orientable.

$U, V \subseteq \mathbb{R}^n$ open, $\varphi : U \xrightarrow{\sim} V$ C^1 diffeo, $\omega \in D^n(V)$, then $\int_V \omega = \pm \int_U \varphi^* \omega$

• We assume now on, to develop the notion of $\int$ that M is oriented.

Cor: $U, V \subseteq \mathbb{R}^n$ open, $\varphi : U \xrightarrow{\sim} V$ orient^n preserving, then $\int_V \omega = \int_U \varphi^* \omega$

→ We extend the above def^n to forms whose support are not nec. contained in a local chart, using partitions of unity.

Def^n: A family of C^r- maps $\{\varphi_\alpha\}_{\alpha \in \Lambda}$, $\varphi_\alpha : M \to \mathbb{R}$

is a C^r- partition of unity subordinate to a covering $\{U_\beta\}_\beta$ of M if

(i) $\mathrm{Supp}\{\varphi_\alpha\} \subseteq U_\beta$. (ii) $\{\mathrm{Supp}\{\varphi_\alpha\}\}_\alpha$ is a locally finite cover of M (i.e. every $p \in M$ has an open nbd that intersects only finitely many of $\mathrm{Supp}\{\varphi_\alpha\}$'s.)

(iii) $$\sum_\alpha \varphi_\alpha(x) = 1 \quad \forall\ x \in M.$$

<u>Thm</u>: For any open covering of a smooth manifold, $\exists$ a partition of unity subordinate to it. $\boxtimes$

For M oriented, let $\mathcal{A}$ be a +ve'ly oriented atlas, ω a top degree form on M. Let $\{f_i\}$ be a smooth partition of unity subordinate to $\mathcal{A}$. Fix +ve'ly oriented coords on the charts in $\mathcal{A}$. We set

$$\int_M \omega := \sum_i \int_{U_i} f_i \omega .$$

→ Each of $\int_{U_i} f_i \omega$ is finite and the sum is over a finite number of indices.